

Euler Characteristics of Crepant Resolutions of Weierstrass Models

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Abstract

Based on an identity of Jacobi, we prove a simple formula that computes the pushforward of analytic functions of the exceptional divisor of a blowup of a projective variety along a smooth complete intersection with normal crossing. We apply this pushforward formula to derive generating functions for Euler characteristics of crepant resolutions of singular Weierstrass models given by Tate's algorithm. Since these Euler characteristics depend only on the sequence of blowups and not on the Kodaira fiber itself, nor the associated group, several distinct Tate models have the same Euler characteristic. In the case of elliptic Calabi-Yau threefolds, we also compute the Hodge numbers. For elliptically fibered Calabi-Yau fourfolds, our results also prove a conjecture of Blumenhagen-Grimm-Jurke-Weigand based on F-theory/heterotic string duality.

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1 Introduction

The study of crepant resolutions of Weierstrass models, their fibral structure, and their flop transitions is an area of common interest to algebraic geometers, number theorists, and string theorists [26–28, 30, 34, 59]. The theory of elliptic surfaces has its beginnings in the 1960s, and was advanced largely by the contributions of mathematicians such as Kodaira [38], Néron [53], Mumford-Suominen [50], Deligne [13], and Tate [60]. Miranda studied the desingularization of elliptic threefolds and the phenomenon of collisions of singularities in [47], and Szydło subsequently generalized Miranda’s work to elliptic n -folds [59]; the Picard number (i.e., the rank of the Néron-Severi group) of an elliptic fibration can be obtained using the Shioda-Tate-Wazir theorem [62]; the study of elliptic fibrations having the same Jacobian was developed by Dolgachev and Gross [17]; and Nakayama studied local and global properties of Weierstrass models over bases of arbitrary dimension in [51, 52]. Furthermore, more recent developments have been inspired by string theory (in particular, M-theory and F-theory) constructions that ascribe an interesting physical meaning to various topological and geometric properties of elliptically-fibered Calabi-Yau varieties [7, 14, 35, 48, 49, 61].

Any elliptic fibration over a smooth base is birational to a (potentially singular) Weierstrass model [13]. Since a Weierstrass model is a hypersurface, it is Gorenstein [19, Corollary 21.19], and hence its canonical class is well-defined as a Cartier divisor. A Weierstrass model $\pi : Y \rightarrow B$ over a smooth base B has at worst \mathbb{Q} -factorial terminal singularities and its canonical class is relatively nef with respect to π . A Weierstrass model provides a friendly framework to discuss several arithmetic invariants such as the Mordell-Weil group and the j -invariant.

In practice, it is often necessary to regularize the singularities of Weierstrass models when computing, for example, their topological invariants. Among the possible regularizations of a singular variety, crepant resolutions are particularly desirable as, by definition, they preserve the canonical class and the smooth locus of the variety. In a sense they modify the variety as mildly as possible while regularizing its singularities. Surfaces with canonical singularities always have a crepant resolution, which is unique up to isomorphism. However, for varieties of dimension three or higher, crepant resolutions do not necessarily exist (they might be obstructed by \mathbb{Q} -factorial terminal singularities), and when they do, they may not be unique. Distinct crepant resolutions of the same Weierstrass model are connected by a network of flops.

Example 1.1. For example, the quadric cone over a conic surface $V(x_1x_2 - x_3x_4) \subset \mathbb{C}^4$ has two crepant resolutions related by an Atiyah flop. The quadric $V(x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2) \subset \mathbb{C}^5$ does not have a crepant resolution since it has \mathbb{Q} -factorial terminal singularities. The binomial variety $V(x_1x_2 - u_1u_2u_3) \subset \mathbb{C}^5$ has six crepant resolutions whose network of flops forms a hexagon [28]. For additional examples involving Weierstrass models, see [23, 24, 26, 27].

There is an important subset of singular Weierstrass models that have crepant resolutions and play a central role in string geometry, as they are instrumental in the geometric engineering of gauge theories in F-theory and M-theory. We refer to them as G -models and define them in §1.2. They are typically defined in connection with Tate’s algorithm [7, 37, 60]. The networks of crepant resolutions of these Weierstrass models are conjectured to have the structure of a hyperplane arrangement that can be defined using simple notions of representation theory [21, 22, 34, 35].

The number of distinct resolutions associated to a G -model can be rather large [21, 22, 34]. It is interesting to study topological invariants that do not depend on the choice of a crepant resolution. An example of such a topological invariant is the Euler characteristic—using p -adic integration and Weil’s conjecture, Batyrev proved that the Betti numbers of smooth varieties connected by a crepant birational map are the same, and it therefore follows that the Euler characteristics of any two crepant resolutions are the same.

The purpose of this paper is to compute the Euler characteristic of G -models obtained by crepant resolutions of Weierstrass models, where G is a simple group. Following [2, 3], we allow the base to be of arbitrary dimension and we do not impose the Calabi-Yau condition. We work relative to a base that we leave arbitrary. In this sense, our paper is a direct generalization of the work of Fullwood and van Hoeij on stringy invariants of Weierstrass models [30].

The Euler characteristic of an elliptic fibration plays a central role in many physical problems such as the computation of gravitational anomalies of six dimensional supergravity theories [32, 54], the cancellation of tadpoles in four dimensional theories [2, 3, 8, 10, 20, 25, 56]. Unfortunately, the Euler characteristics of crepant resolutions of Weierstrass models are generally not known, although they have been computed in some special cases for Calabi-Yau threefolds and fourfolds [4, 5, 30, 45]. For instance, the Euler characteristics of G -models for Calabi-Yau threefolds were studied in [32], and there are conjectures for the Euler characteristic of G -models for Calabi-Yau fourfolds based on heterotic string theory/F-theory duality [8].

As a byproduct of our results, we prove a conjecture by Blumenhagen-Grimm-Jurke-Weigand [8] on the Euler characteristic of Calabi-Yau fourfolds which are G -models for $G = \mathrm{SU}(2)$, $\mathrm{SU}(3)$, $\mathrm{SU}(4)$, $\mathrm{SU}(5)$, E_6 , E_7 or E_8 . These groups correspond to the *exceptional series* E_k defined on page 9 with the exception of D_5 . In [8], the authors conjecture the value of the Euler characteristic using a method inspired by heterotic string theory/F-theory duality. The results of our computation match their prediction precisely, except for the limiting case of the group E_8 . We also retrieve results known for the case of G -models that are Calabi-Yau threefolds [32], while removing most of the assumptions of [32].

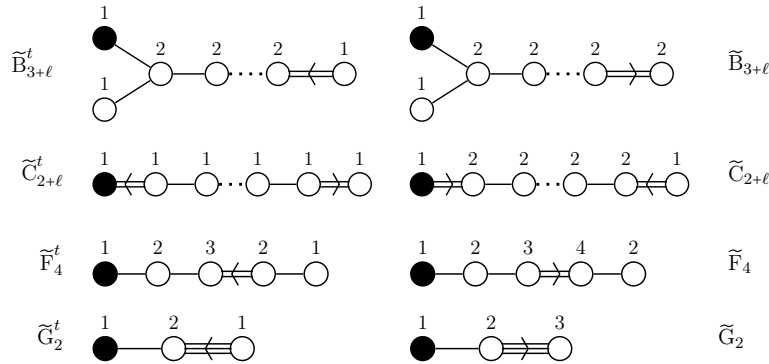
A crucial ingredient of our results is Theorem 1.8, which is a pushforward formula for any analytic function of the class of the exceptional divisor of a blowup, where the center of a blowup is a smooth complete intersection with normal crossing. Theorem 1.8 is very much a direct generalization to arbitrary analytic function of Lemma 2.2 of [30]. Theorem 1.8 profoundly simplifies the algebraic manipulations necessary to compute pushforwards, and is therefore a powerful result quite independently of the specific applications discussed in this paper.

For the reader’s convenience, we provide tables specializing our results to the cases of elliptic threefolds and fourfolds, and further to the cases of Calabi-Yau threefolds and fourfolds, including an explicit computation of the Hodge numbers in the Calabi-Yau threefold case. We emphasize that our results are insensitive to the particular choice of a crepant resolution due to Batyrev’s theorem on the Betti numbers of crepant birational equivalent varieties [6] and Kontsevich’s theorem on the Hodge numbers of birational equivalent Calabi-Yau varieties [39].

1.1 Conventions

Throughout this paper, we work over the field of complex numbers. A variety is a reduced and irreducible algebraic scheme. We mostly follow the notation and conventions of Fulton [31]. Let $\mathcal{V} \rightarrow B$ be a vector bundle over a variety B . We denote by $\mathbb{P}(\mathcal{V})$ the projective bundle of lines in \mathcal{V} . We use Weierstrass models defined with respect to the projective bundle $\pi : X_0 = \mathbb{P}[\mathcal{O}_B \oplus \mathcal{L}^{\otimes 2} \oplus \mathcal{L}^{\otimes 3}] \rightarrow B$ where \mathcal{L} is a line bundle of B . We denote the pullback of \mathcal{L} with respect to π by $\pi^* \mathcal{L}$. We denote by $\mathcal{O}(1)$ the canonical line bundle on X_0 , i.e., the dual of the tautological line bundle of X_0 (see [31, Appendix B.5]). The first Chern class of $\mathcal{O}(1)$ is denoted H and the first Chern class of \mathcal{L} is denoted L . The Weierstrass model $\varphi : Y_0 \rightarrow B$ is defined as the zero-scheme of a section of $\mathcal{O}(3) \otimes \pi^* \mathcal{L}^{\otimes 6}$. Weierstrass models are studied in more detail in §C.3. The Chow group $A_*(X)$ of a nonsingular variety X is the group of divisors modulo rational equivalence [31, Chap. 1, §1.3]. We use $[V]$ to refer to the class of a subvariety V in $A_*(X)$. Given a class $\alpha \in A_*(X)$, the degree of α is denoted $\int_X \alpha$ (or simply $\int \alpha$ if X is clear from the context.) Only the zero component of α is relevant in computing $\int_X \alpha$ —see [31, Definition 1.4, p. 13]. We use $c(X) = c(TX) \cap [X]$ to refer to the total homological Chern class of a nonsingular variety X , and likewise we use $c_i(TX)$ to denote the i th Chern class of the tangent bundle TX . Given two varieties X, Y and a proper morphism $f : X \rightarrow Y$, the proper pushforward associated to f is denoted f_* . If $g : X \rightarrow Y$ is a flat morphism, the pullback of g is denoted g^* and by definition $g^*[V] = [g^{-1}(V)]$, see [31, Chap 1, §1.7]. Given a formal series $Q(t) = \sum_{i=0}^{\infty} Q_i t^i$, we define $[t^n]Q(t) = Q_n$.

Our conventions for affine Dynkin diagrams are as follows. A projective Dynkin diagram is denoted M_k where M is A, B, C, D, E, F , or G , and k is the number of nodes. An affine Dynkin diagram that becomes a projective Dynkin diagram \mathfrak{g} after removing a node of multiplicity one is denoted $\tilde{\mathfrak{g}}$. We denote by $\tilde{\mathfrak{g}}^t$ the (possibly twisted) affine Dynkin diagram whose Cartan matrix is the transpose of the Cartan matrix of $\tilde{\mathfrak{g}}$. The graph of $\tilde{\mathfrak{g}}^t$ is obtained by exchanging the directions of all the arrows of $\tilde{\mathfrak{g}}$. When the extra node is removed, the dual graph of $\tilde{\mathfrak{g}}^t$ reduces to the dual graph of the Langlands dual of \mathfrak{g} . The affine Dynkin diagrams $\tilde{\mathfrak{g}}^t$ and $\tilde{\mathfrak{g}}$ are distinct only when \mathfrak{g} is not simply laced (i.e., when \mathfrak{g} is G_2, F_4, B_k , and C_k). The notation $\tilde{\mathfrak{g}}^t$ follows Carter¹ [9, Appendix, p. 540-609] and is equivalent to the notation $\tilde{\mathfrak{g}}^\vee$ used by MacDonald in §5 of [44]. The multiplicities define a zero vector of the extended Cartan matrix. In the notation of Kac, \tilde{B}_ℓ^t ($\ell \geq 3$), \tilde{C}_ℓ^t ($\ell \geq 2$), \tilde{G}_2^t , and \tilde{F}_4^t are respectively denoted $\tilde{A}_{2\ell-1}^{(2)}$, $\tilde{D}_{\ell+1}^{(2)}$, $\tilde{D}_4^{(3)}$, and $\tilde{E}_6^{(2)}$.



¹There is a typo on page 570 of [9] in the first Dynkin diagram of \tilde{B}_ℓ on the top of the page, where the arrow is in the wrong direction but correctly oriented in the rest of the page.

1.2 G -models

In this section, we recall how a Lie group is naturally associated with an elliptic fibration and introduce the notion of a G -model. Our notation for dual graphs and Kodaira fibers is spelled out in §1.1, and Tables 2 and 3. See also Appendix C for the definitions of a *fiber type*, a *generic fiber*, and a *geometric generic fiber*.

Definition 1.2 (\mathcal{K} -model). Let \mathcal{K} be the type of a generic fiber. Let $S \subset B$ be a divisor of a projective variety B . An elliptic fibration $\varphi : Y \rightarrow B$ over B is said to be \mathcal{K} -model if

1. The discriminant locus $\Delta(\varphi)$ contains as an irreducible component the divisor $S \subset B$.
2. The generic fiber over S is of type \mathcal{K} .
3. Any other fiber away from S is irreducible.

If the dual graph of \mathcal{K} corresponds to an affine Dynkin diagram of type $\tilde{\mathfrak{g}}^t$, where \mathfrak{g} is a Lie algebra, then the \mathcal{K} -model is also called a \mathfrak{g} -model.

In F-theory, a Lie group $G(\varphi)$ attached to a given elliptic fibration $\varphi : Y \rightarrow B$ depends on the type of generic singular fibers and the Mordell-Weil group $\text{MW}(\varphi)$ of the elliptic fibration [12]. The Lie algebra \mathfrak{g} associated to the elliptic fibration is then the Langlands dual $\mathfrak{g}^\vee = \bigoplus_i \mathfrak{g}_i^\vee$ of $\mathfrak{g} = \bigoplus_i \mathfrak{g}_i$. If we denote by $\exp(\mathfrak{g}^\vee)$ the unique (up to isomorphism) simply connected compact group whose Lie algebra is \mathfrak{g}^\vee , then the group associated to the elliptic fibration $\varphi : Y \rightarrow B$ is:

$$G(\varphi) := \frac{\exp(\mathfrak{g}^\vee)}{\text{MW}_{\text{tor}}(\varphi)} \times U(1)^{\text{rk MW}(\varphi)},$$

where $\text{rk MW}(\varphi)$ is the rank of the Mordell-Weil group of φ and $\text{MW}_{\text{tor}}(\varphi)$ is the torsion subgroup of the Mordell-Weil group of φ .

Definition 1.3 (G -model). An elliptic fibration $\varphi : Y \rightarrow B$ with an associated Lie group $G = G(\varphi)$ is called a G -model.

If the reduced discriminant locus has a unique irreducible component S over which the generic fiber is not irreducible, the group $G(\varphi)$ is simple. The relevant fiber $\tilde{\mathfrak{g}}^t$ can be realized by resolving the singularities of a Weierstrass model derived from Tate's algorithm. The relation between the fiber type and the group $G(\varphi)$ is not one-to-one. For example, an $\text{SU}(2)$ -model can be given by a divisor S with a fiber of type I_2^s , I_2^{ns} , III, IV^{ns} , or I_3^{ns} . For that reason, a given decorated Kodaira fiber provides a more refined characterization of a G -model.

Example 1.4. For $n \geq 4$, an $\text{SU}(n)$ -model is a I_n^s -model with a trivial Mordell-Weil group. For $n \geq 0$, a $\text{Spin}(8+2n)$ -model is a $\text{I}_n^{*\text{s}}$ -model with a trivial Mordell-Weil group. For $n \geq 1$, a $\text{Spin}(7+2n)$ -model is an $\text{I}_n^{*\text{ns}}$ -model with a trivial Mordell-Weil group. A G_2 -model is an $\text{I}_0^{*\text{ns}}$ -model with a trivial Mordell-Weil group. A $\text{Spin}(7)$ -model is an $\text{I}_0^{*\text{ss}}$ -model with a trivial Mordell-Weil group.

Example 1.5 (See [24]). The $\text{SO}(3)$, $\text{SO}(5)$, $\text{SO}(6)$, and $\text{SO}(7)$ -models are respectively I_2^{ns} , I_4^{ns} , I_4^s , and $\text{I}_0^{*\text{ss}}$ -models with $\text{MW} = \mathbb{Z}/2\mathbb{Z}$. For $n \geq 0$, an $\text{SO}(8+2n)$ -model is an $\text{I}_n^{*\text{s}}$ -model with a Mordell-Weil group $\text{MW} = \mathbb{Z}/2\mathbb{Z}$. For $n \geq 1$, an $\text{SO}(7+2n)$ -model is an $\text{I}_n^{*\text{ns}}$ -model with Mordell-Weil group $\text{MW} = \mathbb{Z}/2\mathbb{Z}$.

Example 1.6. If the Mordell-Weil group is trivial, \mathcal{K} -models with $\mathcal{K} = \mathbb{I}_2^s, \mathbb{I}_2^{\text{ns}}, \text{III}, \text{IV}^{\text{ns}}, \text{or } \mathbb{I}_3^{\text{ns}}$, are all $\text{SU}(2)$ -models. An A_2 -model can be given by a IV^s -model or a \mathbb{I}_3 -model. If the Mordell-Weil group is trivial, both a IV^s -model or a \mathbb{I}_3^s -model give a $\text{SU}(3)$ -model. A C_ℓ -model can be given by an $\mathbb{I}_{2\ell+2}^{\text{ns}}$ -model or an $\mathbb{I}_{2\ell+3}^{\text{ns}}$ -model. If the Mordell-Weil group is trivial, give a $\text{USp}(2\ell)$ -model.

Remark 1.7. Not all singular Weierstrass models are G -models as the reducible singular fibers might not appear in codimension one. See, for example, the Jacobians of the elliptic fibrations discussed in [2, 3, 20, 25].

1.3 The pushforward theorem and Jacobi's identity

As explained earlier, one of our key results is a pushforward theorem that streamlines all the computations of this paper. We present the pushforward theorem in this subsection.

Theorem 1.8. *Let the nonsingular variety $Z \subset X$ be a complete intersection of d nonsingular hypersurfaces Z_1, \dots, Z_d meeting transversally in X . Let E be the class of the exceptional divisor of the blowup $f : \tilde{X} \rightarrow X$ centered at Z . Let $Q(t) = \sum_a f^* Q_a t^a$ be a formal power series with $Q_a \in A_*(X)$. Then the pushforward $f_* Q(E)$ is:*

$$f_* Q(E) = \sum_{\ell=1}^d Q(Z_\ell) M_\ell, \quad \text{where} \quad M_\ell = \prod_{m \neq \ell} \frac{Z_m}{Z_m - Z_\ell}.$$

We call the coefficient M_ℓ the ℓ -moment of the blowup f .

Remark 1.9. Given a blowup $f : \tilde{X} \rightarrow X$, any element α of the Chow ring $A_*(\tilde{X})$ can be expressed as $\alpha = \sum_{n=0}^\infty f^* \alpha_i E^i$ where α_i are elements of the Chow ring $A_*(X)$. So Theorem 1.8 can be used to pushforward any element of $A_*(\tilde{X})$.

Theorem 1.8 is proven in §3. By the projection formula and the linearity of the pushforward, the proof of Theorem 1.8 is almost trivial once it is established in the special case of a monic monomial $Q(t) = t^k$. This special case is Lemma 3.7 on page 17. The proof of the Lemma 3.7 relies on an identity due to Carl Gustave Jacobi that gives a partial fraction formula for homogeneous complete symmetric polynomials:

Lemma 1.10 (Jacobi). *Let $h_r(x_1, \dots, x_d)$ be the homogeneous complete symmetric polynomial of degree r in d variables of an integral domain. Then:*

$$h_r(x_1, \dots, x_d) = \sum_{\ell=1}^d x_\ell^{r+d-1} \prod_{\substack{m=1 \\ m \neq \ell}}^d \frac{1}{x_\ell - x_m}.$$

Jacobi first proved this identity in 1825 in a slightly different form in his doctoral thesis² as a partial fraction reformulation of the generating function of complete homogeneous polynomials.

² [36, Section III.17, p. 29-30], Jacobi asserts:

$$\prod_i \frac{1}{x - a_i} = \sum_i \frac{1}{x - a_i} \prod_{\ell \neq i} \frac{1}{a_\ell - a_i}$$

Lemma 1.10 was rediscovered in many different mathematical and physical problems, as discussed elegantly in [33]. For example, a proof using Schur polynomials was proposed as the solution to Exercise 7.4 of [58]. For a proof using integrals and residues see Appendix A of [43]; for a proof using matrices, see [11]. We give a short and simple proof of this identity in Appendix A.

We also make use of a second pushforward theorem that concerns the projection from the ambient projective bundle to the base B over which the Weierstrass model is defined. Let \mathcal{V} be a vector bundle of rank r over a nonsingular variety B . The Chow ring of a projective bundle $\pi : \mathbb{P}(\mathcal{V}) \rightarrow B$ is isomorphic to the module $A_*(B)[\zeta]$ modded out by the relation

$$\zeta^r + c_1(\pi^*\mathcal{V})\zeta^{r-1} + \cdots + c_i(\pi^*\mathcal{V})\zeta^{r-i} + \cdots + c_r(\pi^*\mathcal{V}) = 0, \quad \zeta = c_1(\mathcal{O}_{\mathbb{P}(\mathcal{V})}(1)).$$

Theorem 1.11 (See [2, 3, 29]). *Let \mathcal{L} be a line bundle over a variety B and $\pi : X_0 = \mathbb{P}[\mathcal{O}_B \oplus \mathcal{L}^{\otimes 2} \oplus \mathcal{L}^{\otimes 3}] \rightarrow B$ a projective bundle over B . Let $Q(t)$ be a formal power series in t with coefficients in $\pi^*A_*(B)$. Then*

$$\pi_*Q(H) = -2 \left. \frac{Q(H)}{H^2} \right|_{H=-2L} + 3 \left. \frac{Q(H)}{H^2} \right|_{H=-3L} + \frac{Q(0)}{6L^2},$$

where $L = c_1(\mathcal{L})$ and $H = c_1(\mathcal{O}(1))$ is the first Chern class of the dual of the tautological line bundle of X_0 .

Proof. Using the functoriality of Segre classes, we can write

$$\pi_* \left(\frac{1}{1-H} \right) = \frac{1}{(1+2L)(1+3L)} = \frac{-2}{1+2L} + \frac{3}{1+3L},$$

which can be expanded on the both sides. This gives the following expressions for the pushforward of each power of H :

$$\pi_*1 = 0, \quad \pi_*H = 0, \quad \pi_*H^{i+2} = [-2(-2)^i + 3(-3)^i]L^i$$

where i is nonnegative. Then, expanding $Q(H)$ as a power series with coefficients in $A_*(B)$,

$$Q(H) = \sum_{i=0}^{\infty} \pi^*\alpha_i H^i = \pi^*\alpha_0 + (\pi^*\alpha_1)H + H^2 \sum_{k=0}^{\infty} (\pi^*\alpha_k)H^k,$$

the pushforward of $Q(H)$ can hence be computed as

$$\begin{aligned} \pi_*Q(H) &= -2 \sum_{k=0}^{\infty} \alpha_k (-2L)^k + 3 \sum_{k=0}^{\infty} \alpha_k (-3L)^k \\ &= -2 \left. \frac{Q(H) - \alpha_1 H - \alpha_0}{H^2} \right|_{H=-2L} + 3 \left. \frac{Q(H) - \alpha_1 H - \alpha_0}{H^2} \right|_{H=-3L} \\ &= -2 \left. \frac{Q(H)}{H^2} \right|_{H=-2L} + 3 \left. \frac{Q(H)}{H^2} \right|_{H=-3L} + \frac{Q(0)}{6L^2}. \end{aligned}$$

□

1.4 Strategy

We take an intersection theory point of view inspired by Fulton [31] and Aluffi [1], and use explicit crepant resolutions of Tate models to compute their Euler characteristics. Using Chern classes, we evaluate the Euler characteristic without dealing with the combinatorics of the fiber structure. Instead, we compute the pushforward of the homological Chern class of the variety to the base of the fibration. Since the Euler characteristics of two crepant resolutions of the same Weierstrass model are the same [6], we do not need to explore the network of all flops to arrive at our conclusions.

Our method for computing the Euler characteristics of G -models is as follows. Given a choice of Lie group G , we first use Tate's algorithm to determine a singular Weierstrass model $Y_0 \rightarrow B$ with Kodaira fibers associated to G . We then determine a crepant resolution $f : Y \rightarrow Y_0$ of the singular Weierstrass model to obtain an explicit realization of the G -model as a smooth projective variety. By doing so, we retrieve the data necessary to compute the total homological Chern class of the crepant resolution $f : Y \rightarrow Y_0$. We apply Theorem 1.8 repeatedly to push this class forward to the projective bundle X_0 in which the Weierstrass model is defined. Finally, we use Theorem 1.8 to push the total Chern class forward to B . In doing so, we obtain a generating function of the form

$$\chi(Y) = \int_B Q(L, S) c(B), \quad c(B) := c(TB) \cap [B],$$

where \int_B indicates the degree, $Q(L, S)$ is a rational function in L and S such that

$$Q(L, 0) = \frac{12L}{1 + 6L} c(B).$$

$Q(L, 0)$ is the generating function for the Euler characteristic of a smooth Weierstrass model [2]. The rational expression $Q(L, S)c(B)$ is defined in the Chow ring $A_*(B)$ of the base. The expression $\chi(Y)$ is a generating function in the following sense. If the base has dimension d , the Euler characteristic is then given by the coefficient of t^d in a power series expansion in the parameter t :

$$\chi(Y) = [t^d] \left(Q(tL, tS) c_t(TB) \right), \quad \text{where } d := \dim B,$$

where $[t^n]g(t) = g_n$ for a formal series $g(t) = \sum_{i=0}^{\infty} g_i t^i$, and

$$c_t(TB) = 1 + c_1(TB)t + \cdots + c_d(TB)t^d,$$

is the Chern polynomial of the tangent bundle of B .

It follows from the adjunction formula that one can further impose the Calabi-Yau condition by setting $L = c_1(TB)$; see Tables 8 and 9 for the Euler characteristics of elliptic threefold and fourfold G -models.

In Table 1, we organize the Lie algebras associated to our choices of Tate models into a network, where an arrow indicates inclusion as a subalgebra. As is evident from Table 1, the results of this paper cover all instances of Kodaira fibers with the exception of the general cases of I_k and I_k^* that will be discussed in a follow up paper. In particular, our list contains:

$$\begin{array}{ccccccc}
& & C_2 & \longrightarrow & A_3 & \longrightarrow & A_4 & \longrightarrow & D_5 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
A_1 & \longrightarrow & A_2 & \longrightarrow & G_2 & \longrightarrow & B_3 & \longrightarrow & D_4 & \longrightarrow & F_4 & \longrightarrow & E_6 & \longrightarrow & E_7 & \longrightarrow & E_8
\end{array}$$

$I_2^s, I_2^{ns}, III, IV^{ns}, I_3^{ns}$	IV^s, I_3^s	I_4^s	I_4^{ns}	I_5^s	I_0^{*ns}	I_0^{*ss}	I_0^{*s}	IV^{*ns}	I_1^{*s}	IV^{*s}	III^*	II^*
A_1	A_2	A_3	C_2	A_4	G_2	B_3	D_4	F_4	D_5	E_6	E_7	E_8

Table 1: Models studied in this paper.

- G -models corresponding to Deligne exceptional series:

$$\{e\} \subset A_1 \subset A_2 \subset G_2 \subset D_4 \subset F_4 \subset E_6 \subset E_7 \subset E_8.$$

- G -models for the extended exceptional series³:

$$\{e\} \subset A_1 \subset A_2 \subset A_3 \subset E_4 \subset E_5 \subset E_6 \subset E_7 \subset E_8.$$

- G -models for simple orthogonal groups of small rank⁴:

$$\{e\} \subset SO(3) \subset SO(5) \subset SO(6).$$

- G -models of the I_0^* series [23]:

$$\{e\} \subset G_2 \subset Spin(7) \subset Spin(8).$$

1.5 Organization of the paper

The remainder of the paper is organized as follows. In Section 2 we discuss some general properties of the Euler characteristic of elliptic fibrations. In Section 3 we discuss the pushforward theorem and explain the details of our computation of the Euler characteristic. Section 4 then describes how these results can be used to calculate the Hodge numbers of Calabi-Yau threefold G -models. In Section 5, we describe the simplest model, the $SU(2)$ -model, as an example of our computation. We present the results of our computation in a series of tables in Section 6. Finally, in Section 7 we conclude with a discussion of the computation and comment on possible future research directions. A proof of Jacobi's partial fraction identity is given in Appendix A, an explanation of the euler characteristic as the degree of the top Chern class is given in Appendix B, and some basic facts about Kodaira fibers, elliptic fibrations, Weierstrass models and Tate's algorithm are collected in Appendix C.

³ We recall that the Dynkin diagram of E_n is the same as A_n but with the n th node connected with the third node. In particular, $E_4 \cong A_4$, $E_5 \cong D_5$, $E_3 = A_2 \times A_1$, $E_2 = A_2$, and $E_1 = A_1$.

⁴These models require a Mordell-Weil group $\mathbb{Z}/2\mathbb{Z}$; see [24].


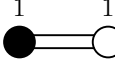
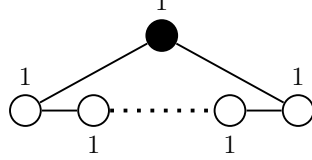
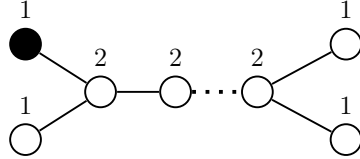
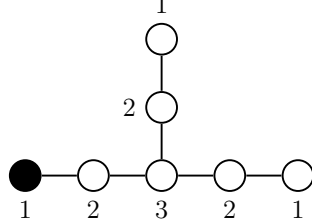
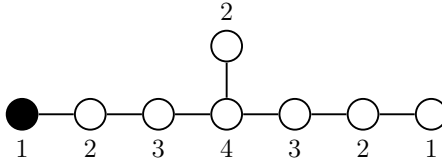
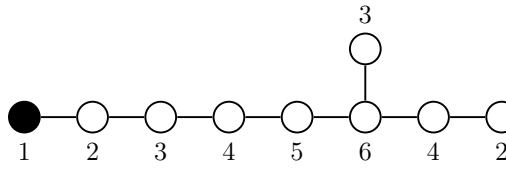
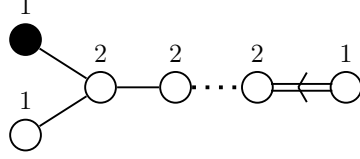
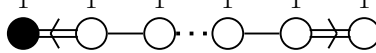
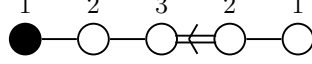
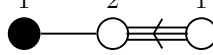
Fiber type	Dynkin diagram	Kodaira type
\tilde{A}_0		I_1, II
\tilde{A}_1		$I_2^s, I_2^{ns}, I_3^{ns}, III, IV^{ns}$
$\tilde{A}_{n-1} \quad (n \geq 3)$		I_n^s
$\tilde{D}_{4+\ell}$		I_ℓ^{*s}
\tilde{E}_6		IV^{*s}
\tilde{E}_7		III^*
\tilde{E}_8		II^{*s}
$\tilde{B}_{3+\ell}^t$		I_ℓ^{*ns}
$\tilde{C}_{2+\ell}^t$		$I_{2\ell+2}^{ns}, I_{2\ell+3}^{ns}$
\tilde{F}_4^t		IV^{*ns}
\tilde{G}_2^t		I_0^{*ns}

Table 2: Affine Dynkin diagrams appearing as dual graphs of decorated Kodaira fibers.

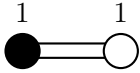
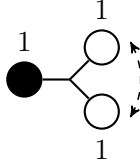
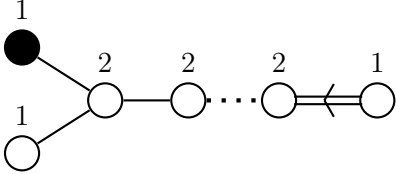
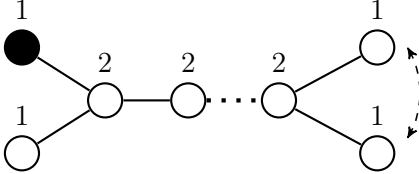
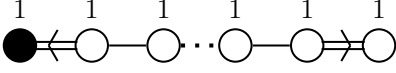
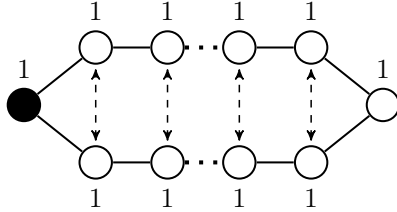
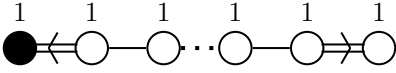
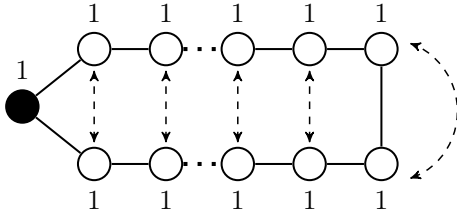
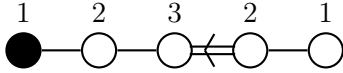
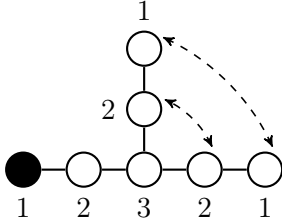
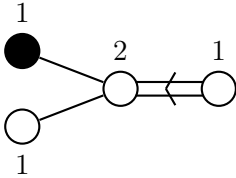
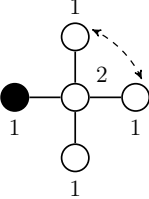
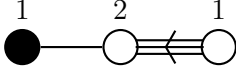
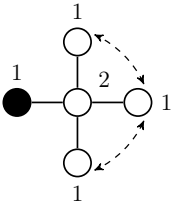
Fiber Type	Dual graph	Dual graph of Geometric fiber
\tilde{A}_1		
$I_{\ell-3}^{*ns}$ \tilde{B}_ℓ^t ($\ell \geq 3$)		
$I_{2\ell+2}^{ns}$ $\tilde{C}_{\ell+1}^t$ ($\ell \geq 1$)		
$I_{2\ell+3}^{ns}$ $\tilde{C}_{\ell+1}^t$ ($\ell \geq 1$)		
IV^{*ns} \tilde{F}_4^t		
I_0^{*ss} \tilde{B}_3^t		
I_0^{*ns} \tilde{G}_2^t		

Table 3: Dual graphs for elliptic fibrations .



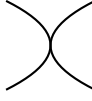

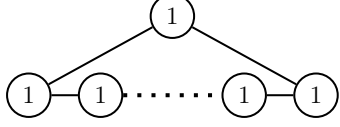
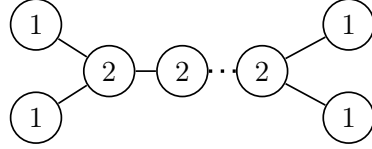
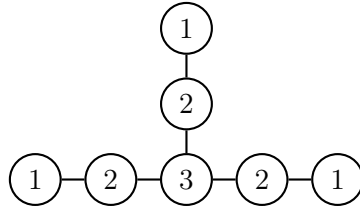
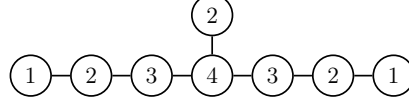
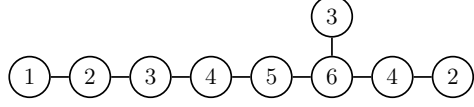
Type	$v(c_4)$	$v(c_6)$	$v(\Delta)$	j	Monodromy	Fiber	Dual Graph
I_0	≥ 0	≥ 0	0	\mathbb{C}	I_2	Smooth	-
I_1	0	0	1	∞	$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$		\tilde{A}_0
II	≥ 1	1	2	0	$\begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}$		\tilde{A}_0
III	1	≥ 2	3	1728	$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$		\tilde{A}_1
IV	≥ 2	2	4	0	$\begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$		\tilde{A}_2
I_n	0	0	$n > 1$	∞	$\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$		\tilde{A}_{n-1}
I_n^*	2	≥ 3	$n+6$	∞	$\begin{pmatrix} -1 & -n \\ 0 & -1 \end{pmatrix}$		\tilde{D}_{n+4}
	≥ 2	3	$n+6$				
IV^*	≥ 3	4	8	0	$\begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}$		\tilde{E}_6
III^*	3	≥ 5	9	1728	$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$		\tilde{E}_7
II^*	≥ 4	5	10	0	$\begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$		\tilde{E}_8

Table 4: Kodaira-Néron classification of geometric fibers over codimension one points of the base of an elliptic fibration [38, 53]. The j -invariant of the I_0^* is never ∞ and can take any finite value.

2 Euler Characteristic of Elliptic Fibrations

In this paper, we confine our attention to the Weierstrass model realization of elliptic fibrations. For more details about this construction, please refer to Section C.3. The Euler characteristic of a smooth Weierstrass model $\varphi: Y \rightarrow B$ over a base B is given by the following formula [2, 3]

$$\chi(Y) = \frac{12L}{1+6L}c(B),$$

where $c(B) = c(TB) \cap [B]$ is the total homological Chern class and $L = c_1(\mathcal{L})$ is the first Chern class of the fundamental line bundle $\mathcal{L} = (R^1\varphi_*\mathcal{O}_X)^{-1}$ of the elliptic fibration. This expression is the generating function for the Euler characteristic. Giving weight to the n th Chern class, the component of pure weight d gives the Euler characteristic of a Weierstrass model over a smooth base B of dimension d . A direct expansion gives

$$\chi(Y) = -2 \sum_{i=1}^d (-6L)^i c_{d-i}(TB) \cap [B].$$

The Euler characteristic of an elliptic surface is given by Kodaira formula [38, III, Theorem 12.2, p. 14]:

$$\chi(Y) = \sum_i v(\Delta_i),$$

where i runs through the points over which the fiber is singular, Δ_i is the minimal discriminant of the singular fiber, and $v(\Delta_i)$ denotes the valuation of Δ_i . In particular, the Euler characteristic of the resolution of a Weierstrass model over a curve is always $12L$:

$$\chi(Y) = 12L.$$

There are in principle several different ways to compute the Euler characteristic of an elliptic fibration. For example, topologically, one can identify the subvarieties V_i of the discriminant locus over which the fiber is constant. The Euler characteristic is then

$$\chi(Y) = \sum_i \chi(V_i) \chi(Y_{\eta_i}),$$

where η_i is the generic point of Δ_i . This method is clearly combinatoric in nature, and increases in complexity with the dimension of the base. For example, the subvarieties D_i are often singular.

A more effective way to compute the Euler characteristic is to use the Poincaré-Hopf Theorem which asserts that the Euler characteristic is given by the integral of the top Chern class. In other words, the Euler characteristic is the degree of the total homological Chern class. This method is explained in Section 2.2 and can also be thought of as an algebraic version of the Chern-Gauss-Bonnet theorem.

2.1 Crepant resolutions and flops

Let X be a projective variety with at worst canonical Gorenstein singularities. We denote the canonical class by K_X .

Definition 2.1. A birational projective morphism $\rho : Y \rightarrow X$ is called a *crepant desingularization* of X if Y is smooth and $K_Y = \rho^* K_X$.

Definition 2.2. A *resolution of singularities* is a birational morphism $f : Y \rightarrow X$ such that Y is smooth, f is proper, and f reduces to an isomorphism away from the singularities of X .

Remark 2.3. A *crepant resolution of singularities* is a resolution of singularities such that $K_Y = f^* K_X$.

Remark 2.4. In dimension two, there is one and only one crepant resolution of a variety with canonical singularities. In dimension three, crepant resolutions of Gorenstein singularities always exist but are usually not unique. In dimension four or greater, crepant resolutions are not always possible. However, one can always find a crepant birational morphism from a \mathbb{Q} -factorial variety with terminal singularities.

Definition 2.5 (*D-flop* (See [46, p. 156-157])). Let $f_1 : X_1 \rightarrow X$ a small contraction. Let D be a \mathbb{Q} -Cartier divisor in X_1 . A *D-flop* is a birational morphism $f : X_1 \dashrightarrow X_2$ fitting into a triangular diagram where f_1 and f_2 are birational morphisms

$$\begin{array}{ccc} X_1 & \xrightarrow{\quad f \quad} & X_2 \\ & \searrow f_1 \quad \swarrow f_2 & \\ & X & \end{array}$$

such that

1. X_i are normal varieties with at worst terminal singularities.
2. f_i are small contractions (i.e. their exceptional loci are in codimension two or higher).
3. K_{X_i} is numerically trivial along the fibers of f_i (i.e. $K_{X_i} \cdot \ell = 0$ for any curve ℓ contracted by f_i).
4. The \mathbb{Q} -divisor $-D$ is f_1 -ample.
5. The strict f -transform D^+ of D is f_2 -ample.

Definition 2.6 (*flop*). The morphism $f_2 : X_2 \rightarrow X$ is said to be a *flop* of $f_1 : X_1 \rightarrow X$ if there exists a divisor $D \subset X_1$ such that f_2 is a D -flop of f_1 .

2.2 Batyrev's theorem and the Chern class of a crepant resolution

We denote the Chow ring of a nonsingular variety X by $A_*(X)$. The free group of classes associated to subvarieties of dimension r modulo rational equivalence is denoted $A_r(X)$. The degree of a class

α of $A_*(X)$ is denoted $\int_X \alpha$ (or simply $\int \alpha$ if there is no ambiguity in the choice of X), and is defined to be the degree of its component in $A_0(X)$. The total homological Chern class $c(X)$ of any nonsingular variety X of dimension d is defined by:

$$c(X) = c(TX) \cap [X],$$

where TX is the tangent bundle of X and $[X]$ is the class of X in the Chow ring. The degree of $c(X)$ is the topological Euler characteristic of X :

$$\chi(X) = \int_X c(X).$$

Batyrev and Dais conjectured that for an algebraic variety with only Gorenstein canonical singularities, the Hodge numbers of a crepant resolution do not depend on the choice of the crepant resolution. Using p -adic integration and the Weil conjecture, Batyrev proved the following slightly weaker proposition:

Theorem 2.7 (Batyrev [6]). *Let X and Y be irreducible birational smooth n -dimensional projective algebraic varieties over \mathbb{C} . Assume that there exists a birational rational map $\varphi : X \dashrightarrow Y$ which does not change the canonical class. Then X and Y have the same Betti numbers.*

Batyrev's result was strongly inspired by string dualities, in particular by the work of Dixon, Harvey, Vafa, and Witten [15]. Kontsevitch proved the Batyrev-Dais conjecture for the special case of Calabi-Yau varieties as a corollary of his newly invented theory of motivic integration; the proof relies on Hodge theory and geometrizes Batyrev's use of p -adic integration.

Theorem 2.8 (Kontsevitch, [39]). *Let X and Y be birationally-equivalent smooth Calabi-Yau varieties. Then X and Y have the same Hodge numbers.*

As a direct consequence of Batyrev's theorem, the Euler characteristic of a crepant resolution of a variety with Gorenstein canonical singularities is independent on the choice of resolution. We identify the Euler characteristic as the degree (see Definition C.2) of the total (homological) Chern class of a crepant resolution $f : \tilde{Y} \rightarrow Y$ of a Weierstrass model $Y \rightarrow B$:

$$\chi(\tilde{Y}) = \int c(\tilde{Y}).$$

We then use the birational invariance of the degree under the pushforward to express the Euler characteristic as a class in the Chow ring of the projective bundle X_0 . We subsequently push this class forward to the base to obtain a rational function depending upon only the total Chern class of the base $c(B)$, the first Chern class $c_1(\mathcal{L})$, and the class S of the divisor in B :

$$\chi(\tilde{Y}) = \int_B \pi_* f_* c(\tilde{Y}).$$

In view of Theorem 2.7, this Euler characteristic is independent of the choice of a crepant resolution. We discuss pushforwards and their role in the computation of the Euler characteristic in more detail in Section 3.

3 Pushforwards and Computing the Euler Characteristic

Definition 3.1 (Pushforward). [[31, Chap. 1, p. 11]] Let $f : X \rightarrow Y$ be a proper morphism. Let V be a subvariety of X , the image $W = f(V)$ a subvariety of Y , and the function field $R(V)$ an extension of the function field $R(W)$. The pushforward $f_* : A_*(X) \rightarrow A_*(Y)$ is defined as follows

$$f_*[V] = \begin{cases} 0 & \text{if } \dim V \neq \dim W, \\ [R(V) : R(W)] [V_2] & \text{if } \dim V = \dim W, \end{cases}$$

where $[R(V) : R(W)]$ is the degree of the field extension $R(V)/R(W)$.

Lemma 3.2 ([31, Chap. 1, p. 13]). *Let $f : X \rightarrow Y$ be a proper map between varieties. For any class α in the Chow ring $A_*(X)$ of X :*

$$\int_X \alpha = \int_Y f_* \alpha.$$

Lemma 3.2 means that an intersection number in X can be computed in Y through a pushforward. This simple fact has far-reaching consequences and defines the point of view taken in this paper, as it allows us to express the topological invariants of an elliptic fibration in terms of those of the base.

3.1 The pushforward theorem

A formula for the Chern classes of blowups of a smooth variety along a smooth center was conjectured by Todd and Segre and proven in the general case by Porteous [55] using the Riemann-Roch theorem. A proof using Riemann-Roch “without denominators” is presented in §15.4 of [31]. A proof without Riemann-Roch was derived by Lascu and Scott [40,41]. A generalization of the formula to potentially singular varieties was obtained by Aluffi [1].

The blowup formula simplifies dramatically when the center of the blowup is a nonsingular complete intersection of nonsingular hypersurfaces meeting transversally. Aluffi gives an elegant short proof using functorial properties of Chern classes and Chern classes of bundles of tangent fields with logarithmic zeros:

Theorem 3.3 (Aluffi, [1, Lemma 1.3]). *Let $Z \subset X$ be the complete intersection of d nonsingular hypersurfaces Z_1, \dots, Z_d meeting transversally in X . Let $f : \tilde{X} \rightarrow X$ be the blowup of X centered at Z . We denote the exceptional divisor of f by E . The total Chern class of \tilde{X} is then:*

$$c(T_{\tilde{X}}) = (1 + E) \left(\prod_{i=1}^d \frac{1 + f^* Z_i - E}{1 + f^* Z_i} \right) f^*(T_X).$$

Lemma 3.4. *Let $f : \tilde{X} \rightarrow X$ be the blowup of X centered at Z . We denote the exceptional divisor of f by E . Then*

$$f_* E^n = (-1)^{d+1} h_{n-d}(Z_1, \dots, Z_d) Z_1 \cdots Z_d,$$

where $h_i(x_1, \dots, x_k)$ is the complete homogeneous symmetric polynomial of degree i in (x_1, \dots, x_k) with the convention that h_i is identically zero for $i < 0$ and $h_0 = 1$.

Proof. The exceptional locus of the blowup of X centered at Z is the projective bundle $\mathbb{P}(N_X Z)$. Let $E = c_1(\mathcal{O}_{\mathbb{P}(N_X Z)}(1))$. By the functoriality of Segre classes, we have:

$$f_* \frac{1}{1+E} \cap [E] = \frac{1}{c(N_X Z)} \cap [Z] = \prod_{i=1}^d \frac{Z_i}{1+Z_i}, \quad (3.1)$$

where $N_X Z$ is the normal bundle of Z in X . The generating function of complete homogeneous symmetric polynomials in (x_1, \dots, x_d) is $\prod_{\ell=1}^d (1 - x_\ell t)^{-1}$:

$$\sum_{n=1}^{\infty} h_n(x_1, \dots, x_d) t^n = \prod_{\ell=1}^d \frac{1}{1 - x_\ell t}.$$

By matching terms of the same dimensions in equation (3.1), we can compute $f_* E^n$ in terms of a complete homogeneous symmetric polynomial in the classes Z_i :

$$f_* E^n = (-1)^{n-1} [t^n] \left(\prod_{i=1}^d \frac{t Z_i}{1 + t Z_i} \right) = (-1)^{d+1} h_{n-d}(Z_1, \dots, Z_d) Z_1 \cdots Z_d,$$

where $[t^n]g(t) = g_n$ for a formal series $g(t) = \sum_{i=0}^{\infty} g_i t^i$. □

Example 3.5. If $d = 2$, we have

$$f_* E = 0, \quad f_* E^2 = -Z_1 Z_2, \quad f_* E^3 = -(Z_1 + Z_2) Z_1 Z_2, \quad f_* E^4 = -(Z_1^2 + Z_2^2 + Z_1 Z_2) Z_1 Z_2.$$

Example 3.6. If $d = 3$, we have

$$f_* E = 0, \quad f_* E^2 = 0, \quad f_* E^3 = Z_1 Z_2 Z_3, \quad f_* E^4 = (Z_1 + Z_2) Z_1 Z_2 Z_3.$$

A direct consequence of Theorem A.2 (Jacobi's identity) and Lemma 3.4 is the following push-forward formula (see for example [30]):

Lemma 3.7. *Let $Z \subset X$ be the complete intersection of d nonsingular hypersurfaces Z_1, \dots, Z_d meeting transversally in X . Let $f : \tilde{X} \rightarrow X$ be the blowup of X centered at Z with exceptional divisor E . Then:*

$$f_* E^n = \sum_{\ell=1}^d Z_\ell^n M_\ell, \quad M_\ell = \prod_{\substack{m=1 \\ m \neq \ell}}^d \frac{Z_m}{Z_m - Z_\ell}.$$

The coefficient M_ℓ is the ℓ -moment of the blowup f defined after Theorem 1.8.

Proof.

$$\begin{aligned}
f_* E^n &= (-1)^{d+1} h_{n-d}(Z_1, \dots, Z_d) Z_1 \cdots Z_d && \text{(by Lemma 3.4)} \\
&= (-1)^{d+1} \sum_{\ell=1}^d Z_\ell^{n-1} \left(\prod_{\substack{m=1 \\ m \neq \ell}}^d \frac{1}{Z_\ell - Z_m} \right) Z_1 \cdots Z_d && \text{(by Lemma 1.10)} \\
&= (-1)^{d+1} \sum_{\ell=1}^d Z_\ell^n \left(\prod_{\substack{m=1 \\ m \neq \ell}}^d \frac{Z_m}{Z_\ell - Z_m} \right) && \text{(by the identity } Z_1 \cdots Z_d = Z_\ell \prod_{\substack{m=1 \\ m \neq \ell}}^d Z_m) \\
&= \sum_{\ell=1}^d Z_\ell^n \left(\prod_{\substack{m=1 \\ m \neq \ell}}^d \frac{Z_m}{Z_m - Z_\ell} \right) && \text{(since } \prod_{\substack{m=1 \\ m \neq \ell}}^d \frac{Z_m}{Z_\ell - Z_m} = (-1)^{d-1} \prod_{\substack{m=1 \\ m \neq \ell}}^d \frac{Z_m}{Z_m - Z_\ell})
\end{aligned}$$

□

To compute topological invariants of a blowup, we often have to pushforward analytic expressions of E . Let $Q(t) = \sum_a f^* Q_a t^a$ be a formal power series with $Q_a \in A_*(X)$. The formal series $Q(E)$ is a well-defined element of $A_*(\tilde{X})$. We recall Theorem 1.8:

Theorem 1.8. *Let the nonsingular variety $Z \subset X$ be a complete intersection of d nonsingular hypersurfaces Z_1, \dots, Z_d meeting transversally in X . Let E be the class of the exceptional divisor of the blowup $f : \tilde{X} \rightarrow X$ centered at Z . Let $Q(t) = \sum_a f^* Q_a t^a$ be a formal power series with $Q_a \in A_*(X)$. Then the pushforward $f_* Q(E)$ is:*

$$f_* Q(E) = \sum_{\ell=1}^d Q(Z_\ell) M_\ell, \quad \text{where} \quad M_\ell = \prod_{m \neq \ell} \frac{Z_m}{Z_m - Z_\ell}.$$

Proof.

$$f_* Q(E) = f_* \sum_a (f^* Q_a) E^a = \sum_a Q_a f_* E^a = \sum_a Q_a \sum_{\ell=1}^d Z_\ell^a M_\ell = \sum_a \sum_{\ell=1}^d Q_a Z_\ell^a M_\ell = \sum_{\ell=1}^d Q(Z_\ell) M_\ell. \quad (3.2)$$

□

3.2 Classes of the blowup centers of crepant resolutions

We denote the projective bundle of the Weierstrass model to be $X_0 = \mathbb{P}[\mathcal{O}_B \oplus \mathcal{L}^{\otimes 2} \oplus \mathcal{L}^{\otimes 3}]$ and the elliptic fibration $\varphi : Y_0 \rightarrow B$ to be the zero-scheme of a section of $\mathcal{O}(3) \otimes \pi^* \mathcal{L}^{\otimes 6}$. We denote by $\mathcal{O}(1)$ the dual of the tautological line bundle of X_0 . We denote by H the first Chern class of $\mathcal{O}(1)$, and by L the first Chern class of \mathcal{L} . The elliptic fibration $\varphi : Y_0 \rightarrow B$ is of class $[Y_0] = 3H + 6\pi^* L$. The classes of the generators of the blowup centers are $Z_i^{(n)}$, where n is the number of the blowup map and i is the number of the center. For example, consider the following blowup:

$$X_0 \xleftarrow{(x, y, s|e_1)} X_1 \xleftarrow{(y, e_1|e_2)} X_2$$

where each arrow above denotes a blowup, and where $E_n = V(e_n)$ is the exceptional divisor of the n th blowup. The first exceptional divisor is a projective bundle whose fibers have projective coordinates $[x' : y' : s']$, where

$$x = x'e_1, \quad y = y'e_1, \quad z = z'e_1.$$

For notational convenience, we drop the prime superscripts $(')$ appearing after each blowup.

The classes associated to the center of the first blowup in (??) are:

$$Z_1^{(1)} = [x] = H + 2\pi^*L, \quad Z_2^{(1)} = [y] = H + 3\pi^*L, \quad Z_3^{(1)} = [s] = \pi^*S.$$

Likewise, the classes associated to the center of the second blowup are

$$Z_1^{(2)} = [y] = f_1^*(H + 3\pi^*L) - E_1, \quad Z_2^{(2)} = [e_1] = E_1.$$

Let us adapt the above data into a matrix-inspired notation, such that i denote columns and n denote rows. This notation allows us to read the classes of the blowup center by each row. In this notation, the above results can be expressed as follows:

$$Z = \begin{pmatrix} Z_1^{(1)} & Z_2^{(1)} & Z_3^{(1)} \\ Z_1^{(2)} & Z_2^{(2)} & \end{pmatrix} = \begin{pmatrix} H + 2\pi^*L & H + 3\pi^*L & \pi^*S \\ f_1^*(H + 3\pi^*L) - E_1 & E_1 & \end{pmatrix}.$$

See Table 6 for an exhaustive list of the generator classes associated to the blowup centers of the crepant resolutions in Table 5. Note that we streamline our notation by omitting the explicit pullback maps from the expressions for the classes appearing in these tables.

4 Hodge Numbers of Elliptically Fibered Calabi-Yau Threefolds

Using motivic integration, Kontsevich shows in his famous ‘‘String Cohomology’’ Lecture at Orsay that birational equivalent Calabi-Yau varieties have the same class in the completed Grothendieck ring [39]. Hence, birational equivalent Calabi-Yau varieties have the same Hodge-Deligne polynomial, Hodge numbers, and Euler characteristic. In this section, we compute the Hodge numbers of crepant resolutions of Weierstrass models in the case of Calabi-Yau threefolds.

Theorem 4.1 (Kontsevich). *Let X and Y be birational equivalent Calabi-Yau varieties over the complex numbers. Then X and Y have the same Hodge numbers.*

Remark 4.2. In Kontsevich’s theorem, a Calabi-Yau variety is a nonsingular complete projective variety of dimension d with a trivial canonical divisor. To compute Hodge numbers in this section, we use the following stronger definition of a Calabi-Yau variety.

Definition 4.3. A *Calabi-Yau variety* is a smooth compact projective variety Y of dimension n with a trivial canonical class and such that $H^i(Y, \mathcal{O}_Y) = 0$ for $1 \leq i \leq n - 1$.

We first recall some basic definitions and relevant classical theorems.

Definition 4.4. The *Néron-Severi group* $\text{NS}(X)$ of a variety X is the group of divisors of X modulo algebraic equivalence. The rank of the Néron-Severi group of X is called the *Picard number* and is denoted $\rho(X)$.

Theorem 4.5 (Lefschetz (1,1)-Theorem). *If X is compact and Kähler, then the map $c_1 : \text{Pic}(X) \rightarrow H^{1,1}(X, \mathbb{Z}) = H^{1,1}(X, \mathbb{C}) \oplus H^2(X, \mathbb{Z})$ is well-defined and surjective. In particular, the Picard number is $\rho(X) = h^{1,1}(X, \mathbb{Z})$.*

Theorem 4.6 (Noether's formula). *If B is a smooth compact, connected, complex surface with canonical class K_B and Euler number c_2 :*

$$\chi(\mathcal{O}_B) = 1 - h^{0,1}(B) + h^{0,2}(B), \quad \chi(\mathcal{O}_B) = \frac{1}{12}(K^2 + c_2).$$

When B is a smooth compact rational surface, we have a simple expression of $h^{1,1}(B)$ as a function of K^2 using the following lemma.

Lemma 4.7. *Let B be a smooth compact rational surface with canonical class K . Then*

$$h^{1,1}(B) = 10 - K^2. \quad (4.1)$$

Proof. Since B is a rational surface, $h^{0,1}(B) = h^{0,2}(B) = 0$. Hence $c_2 = 2 + h^{1,1}$ and the lemma follows from Noether's formula. \square

We now compute $h^{1,1}(Y)$ using the Shioda-Tate-Wazir theorem.

Theorem 4.8 (Shioda-Tate-Wazir; see Corollary 4.1. of [62]). *Let $\varphi : Y \rightarrow B$ be a smooth elliptic threefold, then*

$$\rho(Y) = \rho(B) + f + \text{rank}(MW(\varphi)) + 1$$

where f is the number of geometrically irreducible fibral divisors not touching the zero section.

Theorem 4.9. *Let Y be a smooth Calabi-Yau variety elliptically fibered over a smooth variety B with Mordell-Weil group of rank zero. Then,*

$$h^{1,1}(Y) = h^{1,1}(B) + f + 1, \quad h^{2,1}(Y) = h^{1,1}(Y) - \frac{1}{2}\chi(Y),$$

where f is the number of geometrically irreducible fibral divisors not touching the zero section. In particular, if Y is a G -model with G a simple group, f is the rank of G .

Proof. In the statement of the Shioda-Tate-Wazir theorem, we can replace the Picard numbers $\rho(Y)$ and $\rho(B)$ by the Hodge numbers $h^{1,1}(Y)$ and $h^{1,1}(B)$ using Lefschetz's (1,1)-theorem. That gives $h^{1,1}(Y) = h^{1,1}(B) + f + 1$. Since the Euler characteristic of a Calabi-Yau threefold is $\chi(Y) = 2(h^{1,1} - h^{2,1})$, and assuming that both $\chi(Y)$ and $h^{1,1}(Y)$ are known, it follows that $h^{2,1}(Y) = h^{1,1}(Y) - \frac{1}{2}\chi(Y)$. \square

Remark 4.10. For G -models with G a simple group, f will be the rank of G .

5 An Illustrative Example: SU(2)-Models

In this section, we discuss in detail the computation of the Euler characteristic of SU(2)-models. Note that the results presented in this section are equivalent for each of the four possible Kodaira fibers

(namely, types I_2^s , I_2^s , III, IV^{ns}) realizing an $SU(2)$ -model; see Section 6 for a list of the Weierstrass equations defining the various $SU(2)$ -models. We find

$$c(X_0) = (1 + H)(1 + H + 3\pi^*L)(1 + H + 2\pi^*L)c(B) \quad (5.1)$$

$$c(Y_0) = (3H + 6\pi^*L) \frac{c(X_0)}{1 + 3H + 6\pi^*L}. \quad (5.2)$$

The singular elliptic fibration is resolved by a unique blowup with center (x, y, s) ; see also Table . We denote the blowup as $f : X_1 \rightarrow X_0$ between the ambient spaces and the exceptional divisor is E_1 . In this case, the classes of the center are

$$Z_1 = 3\pi^*L + H, \quad Z_2 = 2\pi^*L + H, \quad Z_3 = \pi^*S. \quad (5.3)$$

The proper transform of Y_0 is denoted Y , and is obtained from the total transform of Y by removing $2E_1$. It follows that the class of Y in X_1 is

$$[Y] = [f^*(3H + 6\pi^*L) - 2E_1] \cap [X_1]$$

Moreover, we have the following Chern classes:

$$c(TX_1) = (1 + E_1) \frac{(1 + f^*Z_1 - E_1)(1 + f^*Z_2 - E_1)(1 + f^*Z_3 - E_1)}{(1 + f^*Z_1)(1 + f^*Z_2)(1 + f^*Z_3)} f^*c(TX_0) \quad (5.4)$$

$$c(TY) = \frac{(1 + E_1)(1 + f^*Z_1 - E_1)(1 + f^*Z_2 - E_1)(1 + f^*Z_3 - E_1)}{(1 + 3H + 6L - 2E_1)(1 + f^*Z_1)(1 + f^*Z_2)(1 + f^*Z_3)} f^*c(TX_0) \quad (5.5)$$

By an expansion of $c(TY)$ in first order, we can easily check that the resolution is crepant:

$$c(TY) = f^*c(TY_0).$$

After the blowup, the homological total Chern class is $c(Y) = c(TY) \cap [Y]$:

$$c(Y) = (3f^*H + 6f^*\pi^*L - 2E_1)(1 + E_1) \frac{(1 + f^*Z_1 - E_1)(1 + f^*Z_2 - E_1)(1 + f^*Z_3 - E_1)}{(1 + f^*Z_1)(1 + f^*Z_2)(1 + f^*Z_3)} f^*c(X_0). \quad (5.6)$$

To compute the Euler characteristic, we have to evaluate

$$\chi(Y) = \int_Y c(Y).$$

The first pushforward requires the the following data:

$$M_1 = \frac{Z_2Z_3}{(Z_2 - Z_1)(Z_3 - Z_1)}, \quad M_2 = \frac{Z_1Z_3}{(Z_1 - Z_2)(Z_3 - Z_2)}, \quad M_3 = \frac{Z_1Z_2}{(Z_1 - Z_3)(Z_2 - Z_3)}. \quad (5.7)$$

Applying the pushforward theorem is now a purely algebraic routine that can be easily implemented in your favorite algebraic software. We pushforward $c(Y)$ defined in the Chow ring of X_1 to the Chow ring of X_0 . After pushing forward to $A_*(X_0)$ using Theorem 1.8, we can apply Theorem 1.11 to compute the pushforward to $A_*(B)$. When the dust settles, we find an expression of $\chi(Y)$ in the

Chow ring of the base:

$$\chi(Y) = \int_Y c(TY) = \int_{X_0} f_* c(TY) = \int_B \pi_* f_* c(TY) = \int_B 6 \frac{2L + 3LS - S^2}{(1+S)(1+6L-2S)} c(TB).$$

Concretely, we replace $c(TB)$ by the Chern polynomial $c_t(TB) = 1 + c_1 t + c_2 t^2 + c_3 t^3 + \dots$, L by Lt , and S by St ; if d is the dimension of B , the Euler characteristic of Y is given by the coefficient of t^d in the Taylor expansion centered at $t = 0$ of the generating function:

$$\begin{aligned} \chi(Y) &= 6 \frac{2Lt + 3LSt^2 - S^2 t^2}{(1+St)(1+6Lt-2St)} c_t(TB) \\ &= 12Lt + 6t^2(2c_1 L - 12L^2 + 5LS - S^2) + 6t^3(-12c_1 L^2 + 5c_1 LS - c_1 S^2) + \dots \end{aligned}$$

Theorem 5.1. *If B is a curve, the Euler characteristic of a $SU(2)$ model is $12L$. If B is a surface, the Euler characteristic is $6(2c_1 L - 12L^2 + 5LS - S^2)$. If B is a threefold, the Euler characteristic is $6(-12c_1 L^2 + 5c_1 LS - c_1 S^2)$.*

In order to consider the Calabi-Yau case, we set $L = c_1(TB)$ in the above expression, which gives

$$\chi(Y) = 12c_1 t - 6t^2(10c_1^2 - 5c_1 S + S^2) + 6t^3(60c_1^3 - 49c_1^2 S + 2c_1 c_2 + 14c_1 S^2 - S^3) + \dots$$

Note that we retrieve the result for a smooth Weierstrass model if we further impose $S = 0$. As a byproduct of the computation of the Euler characteristic of the resolution, we can also easily evaluate the contribution from the singularities to be

$$6 \frac{2L + 3LS - S^2}{(1+S)(1+6L-2S)} c(TB) - \frac{12}{1+6L} c(TB) = 6 \frac{(6L^2 - 2LS + 5L - S)S}{(1+6L)(1+6L-2S)(1+S)} c(TB),$$

which can be rewritten as

$$\chi(Y) - \chi(Y_0) = 6 \frac{6L^2 - 2LS + 5L - S}{(1+6L)(1+6L-2S)} c(S), \quad c(S) = \frac{S}{1+S} c(TB) \cap [B].$$

In the Calabi-Yau case $L = c_1(TB)$, the above quantity usually has a physical meaning. For example, if Y is a Calabi-Yau fourfold, this expression reduces to $-6S(7c_1 - S)^2 \cap [B]$, which is the contribution of branes to the Euler characteristic. In another limit, the above expression can be understood as the contribution of the G_4 -flux in M-theory to the M2-brane flux or brane flux in type IIB string theory:

$$\frac{1}{2} \int_{Y_0} G_4 \wedge G_4 = \frac{1}{2} \int_S F \wedge F = -6 \int_S (7c_1 - S)^2.$$

6 Tables of Results

The G -models studied in this paper are all realized as crepant resolutions of the singular Weierstrass model

$$y^2z + a_1xyz + a_3yz^2 - (x^3 + a_2x^2z + a_4xz^2 + a_6z^3) = 0,$$

where the desired singularity structures corresponding to the decorated Kodaira fibers can be specified by the valuation of the coefficient with respect to the section s whose zero locus is the divisor $S = V(s)$. Following Tate's algorithm, we use the notation $a_{i,p} = a_i/s^p$, where the valuations p are the minimal values dictated by Tate's algorithm and we assume that the coefficients $a_{i,p}$ are generic.

We present the results of our computation of the Euler characteristic generating functions for various G -models. The generating functions are the pushforwards of the homological total Chern class of the resolved Weierstrass model to the base B , and are expressed as rational functions of the classes S and L (where $L = c_1(\mathcal{L})$ is the class of the fundamental line bundle and S is the class of the divisor in the base B), multiplied by the total Chern class of the base, $c(B)$ —see Table 7. Tables 8-10 specialize the results to (respectively) elliptic threefolds, fourfolds, and elliptic Calabi-Yau fourfolds, while Table 11 summarizes the Hodge numbers for Calabi-Yau threefold G -models.

When computing Hodge numbers of a G model which is a Calabi-Yau threefold, we recall that we assume that the base is a rational surface. This is a direct consequence of Definition 4.3. Moreover, for a G -model with G a simple group, the integer f that enters in Theorem 4.9 is the rank of G .

For the $\mathrm{SO}(3)$, $\mathrm{SO}(5)$, and $\mathrm{SO}(6)$ -models, the class S is given by [24]:

$$\begin{cases} S = 4L & \text{for } \mathrm{SO}(3), \\ S = 2L & \text{for } \mathrm{SO}(5), \\ S = 2L & \text{for } \mathrm{SO}(6). \end{cases}$$

Below we list the various Weierstrass equations we use to compute the G -models, labeled by their Kodaira fiber type and associated Lie group G . It is necessary to specify a crepant resolution in order to actually compute the total Chern class and Euler characteristic of a G -model. There could be several distinct crepant resolutions for a G -model. However, Theorem 2.8 assures that the Euler characteristic is insensitive to the choice of crepant resolution and therefore we only need one crepant resolution to compute the Euler characteristic of a G -model defined by the crepant resolution of a Weierstrass model. The models associated to the groups $\mathrm{SU}(n)$ and $\mathrm{USp}(2n)$ are [37]:

$$\mathrm{I}_2^s \quad \mathrm{SU}(2) \quad : \quad y^2z + a_1xyz + a_{3,1}syz = x^3 + a_{2,1}sx^2z + a_{4,1}sxz^2 + a_{6,2}s^2z^3, \quad (6.1)$$

$$\mathrm{I}_{2n}^{\mathrm{ns}} \quad \mathrm{USp}(2n) \quad : \quad y^2z = x^3 + a_2x^2z + a_{4,n}s^nxz^2 + a_{6,2n}s^{2n}z^3, \quad (6.2)$$

$$\mathrm{I}_{2n+1}^{\mathrm{ns}} \quad \mathrm{USp}(2n) \quad : \quad y^2z = x^3 + a_2x^2z + a_{4,n+1}s^{n+1}xz^2 + a_{6,2n}s^{2n+1}z^3, \quad (6.3)$$

$$\mathrm{I}_{2n}^s \quad \mathrm{SU}(2n) \quad : \quad y^2z + a_1xyz = x^3 + a_{2,1}sx^2z + a_{4,n}s^nxz^2 + a_{6,2n}s^{2n}z^3, \quad (6.4)$$

$$\mathrm{I}_{2n+1}^s \quad \mathrm{SU}(2n+1) \quad : \quad y^2z + a_1xyz + a_{3,n}s^nyz^2 = x^3 + a_{2,1}sx^2z + a_{4,n+1}s^{n+1}xz^2 + a_{6,2n+1}s^{2n+1}z^3. \quad (6.5)$$

The Weierstrass models for $\text{SO}(3)$, $\text{SO}(5)$, and $\text{SO}(6)$ are discussed in [24]; these models require a Mordell-Weil group $\mathbb{Z}/2\mathbb{Z}$. The crepant resolutions of the Weierstrass models for G_2 , $\text{Spin}(7)$, and $\text{Spin}(8)$ models are studied in [23] and require a careful analysis of the Galois group of an associated polynomial. The Weierstrass equations defining these models as well as the remaining exceptional groups [37] are given below:

$$I_2^{\text{ns}} \quad \text{SO}(3) \quad : \quad y^2 z = x(x^2 + a_2 x z + a_4 z^2), \quad (6.6)$$

$$I_4^{\text{ns}} \quad \text{SO}(5) \quad : \quad y^2 z = (x^3 + a_2 x^2 z + s^2 x z^2), \quad (6.7)$$

$$I_4^{\text{s}} \quad \text{SO}(6) \quad : \quad y^2 z + a_1 x y z = x^3 + m s x^2 z + s^2 x z^2, \quad m \in \mathbb{C}, \quad m \neq -2, 0, 2, \quad (6.8)$$

$$I_0^{*\text{ss}} \quad \text{Spin}(7) \quad : \quad y^2 z = x^3 + a_{2,1} s x^2 z + a_{4,2} s^2 x z^2 + a_{6,4} s^4 z^3, \quad (6.9)$$

$$I_0^{*\text{s}} \quad \text{Spin}(8) \quad : \quad y^2 z = (x - x_1 s z)(x - x_2 s z)(x - x_3 s z) + s^2 r x^2 z + s^3 q x z^2 + s^4 t z^3, \quad (6.10)$$

$$\text{III} \quad \text{SU}(2) \quad : \quad y^2 z = x^3 + s a_{4,1} x z^2 + s^2 a_{6,2} z^3, \quad (6.11)$$

$$\text{IV}^{\text{ns}} \quad \text{SU}(2) \quad : \quad y^2 z = x^3 + s^2 a_{4,2} x z^2 + s^2 a_{6,2} z^3, \quad (6.12)$$

$$\text{IV}^{\text{s}} \quad \text{SU}(3) \quad : \quad y^2 z + a_{3,1} s y z^2 = x^3 + s^2 a_{4,2} x z^2 + s^3 a_{6,3} z^3, \quad (6.13)$$

$$I_0^{*\text{ns}} \quad G_2 \quad : \quad y^2 z = x^3 + s^2 a_{4,2} x z^2 + s^3 a_{6,3} z^3, \quad (6.14)$$

$$\text{IV}^{*\text{ns}} \quad F_4 \quad : \quad y^2 z = x^3 + s^3 a_{4,3} x z^2 + s^4 a_{6,4} z^3, \quad (6.15)$$

$$\text{IV}^{*\text{s}} \quad E_6 \quad : \quad y^2 z + a_{3,2} s^2 y z^2 = x^3 + s^3 a_{4,3} x z^2 + s^5 a_{6,5} z^3, \quad (6.16)$$

$$\text{III}^* \quad E_7 \quad : \quad y^2 z = x^3 + s^3 a_{4,3} x z + s^5 a_{6,5} z^3 \quad (6.17)$$

$$\text{II}^* \quad E_8 \quad : \quad y^2 z = x^3 + s^4 a_{4,4} x z^2 + s^5 a_{6,5} z^3. \quad (6.18)$$

Group	Kodaira Type	Resolution
SU(2)	$I_2^s, I_2^{\text{ns}}, \text{III}, \text{IV}^{\text{ns}}$	$X_0 \xleftarrow{(x, y, s e_1)} X_1$
SU(3) USp(4) G ₂	$I_3^s, \text{IV}^{\text{ns}}$ I_4^{ns} $I_0^{*\text{ns}}$	$X_0 \xleftarrow{(x, y, s e_1)} X_1 \xleftarrow{(y, e_1 e_2)} X_2$
SU(4) Spin(7)	I_4^s I_0^{*ss}	$X_0 \xleftarrow{(x, y, s e_1)} X_1 \xleftarrow{(y, e_1 e_2)} X_2 \xleftarrow{(x, e_2 e_3)} X_3$
Spin(8)	I_0^{*s}	$X_0 \xleftarrow{(x, y, s e_1)} X_1 \xleftarrow{(y, e_1 e_2)} X_2 \xleftarrow{(x - x_i s z, e_2 e_3)} X_3 \xleftarrow{(x - x_j s z, e_2 e_4)} X_4$
F ₄	$\text{IV}^{*\text{ns}}$	$X_0 \xleftarrow{(x, y, s e_1)} X_1 \xleftarrow{(y, e_1 e_2)} X_2 \xleftarrow{(x, e_2 e_3)} X_3 \xleftarrow{(e_2, e_3 e_4)} X_4$
SU(5)	I_4^s	$X_0 \xleftarrow{(x, y, s e_1)} X_1 \xleftarrow{(x, y, e_1 e_2)} X_2 \xleftarrow{(y, e_1 e_3)} X_3 \xleftarrow{(y, e_2 e_4)} X_4$
Spin(10)	I_1^{*s}	$X_0 \xleftarrow{(x, y, s e_1)} X_1 \xleftarrow{(y, e_1 e_2)} X_2 \xleftarrow{(x, e_2 e_3)} X_3 \xleftarrow{(y, e_3 e_4)} X_4 \xleftarrow{(e_2, e_3 e_5)} X_5$
E ₆	IV^{*s}	$X_0 \xleftarrow{(x, y, s e_1)} X_1 \xleftarrow{(y, e_1 e_2)} X_2 \xleftarrow{(x, e_2 e_3)} X_3 \xleftarrow{(e_2, e_3 e_4)} X_4$ $X_6 \xrightarrow{(y, e_4 e_6)} X_5 \uparrow (y, e_3 e_5)$
E ₇	III^*	$X_0 \xleftarrow{(x, y, s e_1)} X_1 \xleftarrow{(y, e_1 e_2)} X_2 \xleftarrow{(x, e_2 e_3)} X_3 \xleftarrow{(y, e_3 e_4)} X_4$ $X_7 \xrightarrow{(e_4, e_5 e_7)} X_6 \xrightarrow{(e_2, e_4 e_6)} X_5 \uparrow (e_2, e_3 e_5)$
E ₈	II^*	$X_0 \xleftarrow{(x, y, s e_1)} X_1 \xleftarrow{(y, e_1 e_2)} X_2 \xleftarrow{(x, e_2 e_3)} X_3 \xleftarrow{(y, e_3 e_4)} X_4$ $X_8 \xrightarrow{(e_4, e_7 e_8)} X_7 \xrightarrow{(e_2, e_4, e_6 e_7)} X_6 \xrightarrow{(e_4, e_5 e_6)} X_5 \uparrow (e_2, e_3 e_5)$
SO(3)	I_2^{ns}	$X_0 \xleftarrow{(x, y e_1)} X_1$
SO(5)	I_4^{ns}	$X_0 \xleftarrow{(x, y, s e_1)} X_1 \xleftarrow{(x, y, e_1 e_2)} X_2$
SO(6)	I_4^s	$X_0 \xleftarrow{(x, y, s e_1)} X_1 \xleftarrow{(y, e_1 e_2)} X_2 \xleftarrow{(x, e_2 e_3)} X_3$

Table 5: The blowup centers of the crepant resolutions. See the beginning of Section 3.2 for an explanation of our notation.

Algebra	Group	Generator classes of the blowup centers ($Z_i^{(n)}$)
A ₁	SU(2)	$(H + 2L \ H + 3L \ S)$
A ₂ C ₂ G ₂	SU(3) USp(4) G ₂	$\begin{pmatrix} H + 2L & H + 3L & S \\ H + 3L - E_1 & E_1 & \end{pmatrix}$
A ₃	SU(4)	$\begin{pmatrix} H + 2L & H + 3L & S \\ H + 3L - E_1 & E_1 & \\ H + 2L - E_1 & E_2 & \end{pmatrix}$
D ₄	Spin(8)	$\begin{pmatrix} H + 2L & H + 3L & S \\ H + 3L - E_1 & E_1 & \\ H + 2L - E_1 & E_2 & \\ H + 2L - E_1 & E_2 - E_3 & \end{pmatrix}$
F ₄	F ₄	$\begin{pmatrix} H + 2L & H + 3L & S \\ H + 3L - E_1 & E_1 & \\ H + 2L - E_1 & E_2 & \\ E_2 - E_3 & E_3 & \end{pmatrix}$
A ₄	SU(5)	$\begin{pmatrix} H + 2L & H + 3L & S \\ H + 2L - E_1 & H + 3L - E_1 & E_1 \\ H + 3L - E_1 - E_2 & E_1 - E_2 & \\ H + 3L - E_1 - E_2 - E_3 & E_2 & \end{pmatrix}$
D ₅	Spin(10)	$\begin{pmatrix} H + 2L & H + 3L & S \\ H + 3L - E_1 & E_1 & \\ H + 2L - E_1 & E_2 & \\ H + 3L - E_1 - E_2 & E_3 & \\ E_2 - E_3 & E_3 - E_4 & \end{pmatrix}$
E ₆	E ₆	$\begin{pmatrix} H + 2L & H + 3L & S \\ H + 3L - E_1 & E_1 & \\ H + 2L - E_1 & E_2 & \\ E_2 - E_3 & E_3 & \\ H + 3L - E_1 - E_2 & E_3 - E_4 & \\ H + 3L - E_1 - E_2 - E_5 & E_4 & \end{pmatrix}$
E ₇	E ₇	$\begin{pmatrix} H + 2L & H + 3L & S \\ H + 3L - E_1 & E_1 & \\ H + 2L - E_1 & E_2 & \\ H + 3L - E_1 - E_2 & E_3 & \\ E_2 - E_3 & E_3 - E_4 & \\ E_2 - E_3 - E_5 & E_4 & \\ E_4 - E_6 & E_5 & \end{pmatrix}$
E ₈	E ₈	$\begin{pmatrix} H + 2L & H + 3L & S \\ H + 3L - E_1 & E_1 & \\ H + 2L - E_1 & E_2 & \\ H + 3L - E_1 - E_2 & E_3 & \\ E_2 - E_3 & E_3 - E_4 & \\ E_4 & E_5 & \\ E_2 - E_3 - E_5 & E_4 - E_6 & E_6 \\ E_4 - E_6 - E_7 & E_7 & \end{pmatrix}$
A ₁	SO(3)	$(H + 2L \ H + 3L)$
B ₂	SO(5)	$\begin{pmatrix} H + 2L & H + 3L & 2L \\ H + 2L - E_1 & H + 3L - E_1 & E_1 \end{pmatrix}$
A ₃	SO(6)	same as SU(4), with $S = 2L$

Table 6: The classes of the centers of the blowups for all G -models

Algebra	Group	Kodaira Fiber	$\chi(Y) = \pi_* (\psi_* c(TY) \cap [Y])$
–	{e}	I ₁	$\frac{12L}{1+6L} c(B)$
A ₁	SU(2)	I ₂ ^s , I ₂ ^{ns} , III, IV ^{ns}	$6 \frac{2L+3LS-S^2}{(1+S)(1+6L-2S)} c(B)$
A ₂	SU(3)	I ₃ ^s , IV ^s	$12 \frac{L+2SL-S^2}{(1+S)(1+6L-3S)} c(B)$
C ₂	USp(4)	I ₄ ^{ns}	
G ₂	G ₂	I ₀ ^{*ns}	
A ₃	SU(4)	I ₄ ^s	$4 \frac{3L+12L^2+LS-5S^2+30L^2S-35LS^2+10S^3}{(1+S)(1+6L-4S)(1+4L-2S)} c(B)$
A ₂	SU(3)	I ₃ ^s	
C ₂	USp(4)	I ₄ ^{ns}	
G ₂	G ₂	I ₀ ^{*ns}	
B ₃	Spin(7)	I ₀ ^{*ss}	
D ₄	Spin(8)	I ₀ ^{*s}	$12 \frac{L+3SL-2S^2}{(1+S)(1+6L-4S)} c(B)$
F ₄	F ₄	IV ^{*ns}	
A ₄	SU(5)	I ₅ ^s	$\frac{12L+42L^2S+12L^2-35LS^2+32LS-30S^2}{(1+L)(1+S)(-1+6L-5S)} c(B)$
D ₅	Spin(10)	I ₁ ^{*s}	$\frac{4(-8(4L+1)S^2+6(4L+1)LS+3(2L+1)L+10S^3)}{(S+1)(-2L+S-1)(-6L+5S-1)} c(B)$
E ₆	E ₆	IV ^{*s}	$3 \frac{4L+12L^2-12S^2+6SL-81S^2L+54SL^2+30S^3}{(1+S)(1+6L-5S)(1+3L-2S)} c(B)$
E ₇	E ₇	III [*]	$2 \frac{6L+24L^2+7LS-21S^2+120L^2S-190LS^2+75S^3}{(1+S)(1+6L-5S)(1+4L-3S)} c(B)$
E ₈	E ₈	II [*]	$12 \frac{L+6LS-5S^2}{(1+S)(1+6L-5S)} c(B)$
A ₁	SO(3)	I ₂ ^{ns}	$12 \frac{12L}{1+4L} c(B)$
B ₂	SO(5)	I ₄ ^{ns}	$12 \frac{4L(3+4L)}{(1+2L)^2} c(B)$
A ₃	SO(6)	II ₄ ^s	$12 \frac{12L}{1+2L} c(B)$

Table 7: Generating functions of Euler characteristic of crepant resolutions of Tate’s models with trivial Mordell-Weil groups. S is the divisor over which the generic fiber is of type given by the Kodaira fiber and $L = c_1(\mathcal{L})$ where \mathcal{L} is the fundamental line bundle of the Weierstrass model.

Models	$\chi(Y_3)$, Euler characteristic
Smooth Weierstrass	$12L(c_1 - 6L)$
SU(2)	$6(2c_1L - 12L^2 + 5LS - S^2)$
SU(3) or USp(4) or G_2	$12(c_1L - 6L^2 + 4LS - S^2)$
SU(4) or Spin(7)	$4(3c_1L - 18L^2 + 16LS - 5S^2)$
Spin(8) or F_4	$12(c_1L - 6L^2 + 6LS - 2S^2)$
SU(5)	$2(6c_1L - 36L^2 + 40LS - 15S^2)$
Spin(10)	$4(3c_1L - 18L^2 + 21LS - 8S^2)$
E_6	$6(2c_1L - 12L^2 + 15LS - 6S^2)$
E_7	$2(6c_1L - 36L^2 + 49LS - 21S^2)$
E_8	$12(c_1L - 6L^2 + 10LS - 5S^2)$
SO(3)	$12L(c_2 - 4c_1L + 16L^2)$
SO(5)	$12L(20L^2 - 8c_1L + 3c_2)$
SO(6)	$12(4L^2 - 2Lc_1 + c_2)L$

Table 8: Euler characteristic for elliptic threefolds

Models	$\chi(Y_4)$, Euler characteristic
Smooth Weierstrass	$12L(-6c_1L + c_2 + 36L^2)$
SU(2)	$6(-12c_1L^2 + 5c_1LS - c_1S^2 + 2c_2L + 72L^3 - 54L^2S + 15LS^2 - S^3)$
SU(3) or USp(4) or G_2	$12(-6c_1L^2 + 4c_1LS - c_1S^2 + c_2L + 36L^3 - 42L^2S + 17LS^2 - 2S^3)$
SU(4) or Spin(7)	$4(-18c_1L^2 + 16c_1LS - 5c_1S^2 + 3c_2L + 108L^3 - 166L^2S + 89LS^2 - 15S^3)$
SU(5)	$-72c_1L^2 + 80c_1LS - 30c_1S^2 + 12c_2L + 432L^3 - 830L^2S + 555LS^2 - 120S^3$
Spin(10)	$4(-18c_1L^2 + 21c_1LS - 8c_1S^2 + 3c_2L + 108L^3 - 210L^2S + 140LS^2 - 30S^3)$
Spin(8) or F_4	$12(-6c_1L^2 + c_2L + 36L^3 + 6c_1LS - 2c_1S^2 - 60L^2S + 34LS^2 - 6S^3)$
E_6	$3(-24c_1L^2 + 30c_1LS - 12c_1S^2 + 4c_2L + 144L^3 - 288L^2S + 195LS^2 - 42S^3)$
E_7	$2(-36c_1L^2 + 49c_1LS - 21c_1S^2 + 6c_2L + 216L^3 - 454L^2S + 321LS^2 - 72S^3)$
E_8	$12(-6c_1L^2 + 10c_1LS - 5c_1S^2 + c_2L + 36L^3 - 90L^2S + 75LS^2 - 20S^3)$
SO(3)	$12L(c_3 - 4c_2L + 16c_1L^2 - 64L^3)$
SO(5)	$4L(-48L^3 + 20L^2c_1 - 8Lc_2 + 3c_3)$
SO(6)	$12L(-8L^3 + 4L^2c_1 - 2Lc_2 + c_3)$

Table 9: Euler characteristic for elliptic fourfolds

Models	$\chi(Y_4)$, Euler characteristic
Smooth Weierstrass	$12c_1c_2 + 360c_1^3$
SU(2)	$6(2c_1c_2 + 60c_1^3 - 49c_1^2S + 14c_1S^2 - S^3)$
SU(3) or USp(4) or G_2	$12(c_1c_2 + 30c_1^3 - 38c_1^2S + 16c_1S^2 - 2S^3)$
SU(4) or Spin(7)	$12(3c_1c_2 + 30c_1^3 - 50c_1^2S + 28c_1S^2 - 5S^3)$
Spin(8) or F_4	$12(c_1c_2 + 30c_1^3 - 54c_1^2S + 32c_1S^2 - 6S^3)$
SU(5)	$3(4c_1c_2 + 120c_1^3 - 250c_1^2S + 175c_1S^2 - 40S^3)$
Spin(10)	$12(c_1c_2 + 30c_1^3 - 63c_1^2S + 44c_1S^2 - 10S^3)$
E_6	$3(4c_1c_2 + 120c_1^3 - 258c_1^2S + 183c_1S^2 - 42S^3)$
E_7	$6(2c_1c_2 + 60c_1^3 - 135c_1^2S + 100c_1S^2 - 24S^3)$
E_8	$12(c_1c_2 + 30c_1^3 - 80c_1^2S + 70c_1S^2 - 20S^3)$
SO(3)	$12c_1(c_3 - 48c_1^3 + -4c_1c_2)$
SO(5)	$4c_1(3c_3 - 28c_1^3 - 8c_1c_2)$
SO(6)	$12c_1(-4c_1^3 - 2c_1c_2 + c_3)$

Table 10: Euler characteristic for Calabi-Yau elliptic fourfolds where $c_1 = L$.

Algebra	Group	Kodaira Fiber	$h^{1,1}(Y_3)$	$h^{2,1}(Y_3)$	$\chi(Y_3)$
–	{e}	I_1	$11 - K^2$	$11 + 29K^2$	$-60K^2$
A_1	$SU(2)$	$I_2^s, I_2^{ns}, III, IV^{ns}$	$12 - K^2$	$12 + 29K^2 + 15KS + 3S^2$	$-60K^2 - 30KS - 6S^2$
A_2 C_2 G_2	$SU(3)$ $USp(4)$ G_2	I_3^s, IV^s I_4^{ns} I_0^{*ns}	$13 - K^2$	$13 + 29K^2 + 24KS + 6S^2$	$-60K^2 - 48KS - 12S^2$
A_3 B_3	$SU(4)$ $Spin(7)$	I_4^s I_0^{*ss}	$14 - K^2$	$14 + 29K^2 + 32KS + 10S^2$	$-60K^2 - 64KS - 20S^2$
D_4 F_4	$Spin(8)$ F_4	I_0^{*s} IV^{*ns}	$15 - K^2$	$15 + 29K^2 + 36KS + 12S^2$	$-60K^2 - 72KS - 24S^2$
A_4	$SU(5)$	I_5^s	$15 - K^2$	$15 + 29K^2 + 40KS + 15S^2$	$-60K^2 - 80KS - 30S^2$
D_5	$Spin(10)$	I_1^{*s}	$16 - K^2$	$16 + 29K^2 + 42KS + 16S^2$	$-60K^2 - 84KS - 32S^2$
E_6	E_6	IV^{*s}	$17 - K^2$	$17 + 29K^2 + 45KS + 18S^2$	$-60K^2 - 90KS - 36S^2$
E_7	E_7	III^*	$18 - K^2$	$18 + 29K^2 + 49KS + 21S^2$	$-60K^2 - 98KS - 42S^2$
E_8	E_8	II^*	$19 - K^2$	$19 + 29K^2 + 60KS + 30S^2$	$-60K^2 - 120KS - 60S^2$
A_1	$SO(3)$	I_2^{ns}	$12 - K^2$	$12 + 17K^2$	$-36K^2$
B_2	$SO(5)$	I_4^{ns}	$14 - K^2$	$14 + 9K^2$	$-20K^2$
A_3	$SO(6)$	I_4^s	$14 - K^2$	$14 + 5K^2$	$-12K^2$

Table 11: Hodge numbers and Euler characteristic of Calabi-Yau threefolds obtained from crepant resolutions of Tate's models.

7 Discussion

In this paper, we have computed the generating functions for the Euler characteristics of G -models obtained by crepant resolutions of Weierstrass models with bases of arbitrary dimension. The case of G -models that are also Calabi-Yau varieties is important in string theory and is treated here as a special case. In particular, we list the Euler characteristic of G -models that are elliptic threefolds and fourfolds. For Calabi-Yau threefolds, we also compute the Hodge numbers. These results are insensitive to the particular choice of resolution due to Batyrev's theorem on the Betti numbers of crepant birational equivalent varieties and Kontsevich's theorem on the Hodge numbers of birational equivalent Calabi-Yau varieties [6, 39]. We have considered all possible G -models with G a simple Lie group, except for the case of Kodaira fibers $I_{n>5}$ and $I_{n>1}^*$ that we will treat in a follow-up paper.

Given a G -model given by a singular Weierstrass model

$$\varphi : Y_0 \longrightarrow B$$

with crepant resolution $f : Y \longrightarrow Y_0$. The Weierstrass equation is defined as a hypersurface in the projective bundle

$$\pi : \mathbb{P}[\mathcal{O}_B \oplus \mathcal{L}^{\otimes 2} \oplus \mathcal{L}^{\otimes 3}] \rightarrow B,$$

where \mathcal{L} is the fundamental line bundle of the Weierstrass model. In the Calabi-Yau case, $c_1(\mathcal{L}) = c_1(TB)$. We compute the Euler characteristic of Y as the degree of its total Chern class defined in homology

$$\chi(Y) = \int_Y c(Y).$$

We work relative to an arbitrary base B of arbitrary dimension. Using the functorial properties of the degree, we pushforward first to the Chow ring of the projective bundle and then to the Chow ring of the base:

$$\chi(Y) = \int_B \pi_* f_* c(Y).$$

The final result is a generating function for the Euler characteristic.

A key result of this work is Theorem 1.8, which has numerous applications in intersection theory. We also provide a simple proof of an identity (Lemma 1.10) that can be traced back to Jacobi's thesis and appears in numerous situations in mathematics and physics, which is instrumental in the proof of Theorem 1.8.

We also retrieve in a unifying way known results on the Euler characteristics and Hodge numbers of Calabi-Yau threefolds. Furthermore, we have proven *en passant* a conjecture of Blumenhagen-Grimm-Jurke-Weigand [8] on the Euler characteristics of Calabi-Yau fourfolds that are G -models with G belonging to the exceptional series. One interesting point that is almost trivial from the perspective taken in this paper is that certain G -models with different G will have the same Euler characteristic just because they are resolved by the same sequence of blowups.

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A Jacobi’s Partial Fraction Identity

In this section, we prove a formula of Jacobi and exploit the theorem to give a simple proof of a formula of Louck and Biedenharn [43, Appendix A, p. 2400] by demonstrating its equivalence with the following theorem of Jacobi.

Theorem A.1 (Jacobi [36, Section III.17, p. 29-30]). *Let a_i ($i = 1, \dots, d$) be d distinct elements of an integral domain. Then*

$$\prod_{i=1}^d \frac{1}{x - a_i} = \sum_{i=1}^d \frac{1}{x - a_i} \prod_{\substack{j=1 \\ j \neq i}}^d \frac{1}{a_i - a_j}. \quad (\text{A.1})$$

Proof. Let

$$F(x) = \prod_{i=1}^d \frac{1}{x - a_i}, \quad (\text{A.2})$$

where $a_i \neq a_j$ for $i \neq j$. We would like to find the partial fraction expansion of $F(x)$. That is, we would like to find coefficients A_i ($i = 1, \dots, d$) such that

$$F(x) = \sum_{i=1}^d \frac{A_i}{x - a_i}. \quad (\text{A.3})$$

We determine A_i by the *method of residues*. Multiplying (A.3) by $(x - a_i)$, simplifying, and evaluating at $x = a_i$ gives

$$(x - a_j)F(x)|_{x=a_j} = A_j.$$

Applying the above formula to (A.2), we get $A_j = \prod_{i \neq j} \frac{1}{a_i - a_j}$, which is the identity of Jacobi:

$$\prod_{i=1}^d \frac{1}{x - a_i} = \sum_{i=1}^d \frac{1}{x - a_i} \prod_{\substack{j=1 \\ j \neq i}}^d \frac{1}{a_i - a_j}. \quad (\text{A.4})$$

□

Theorem A.2 (Jacobi, Louck-Biedenharn, Cornelius). *Let $h_r(x_1, \dots, x_d)$ be the homogeneous complete symmetric polynomial of degree r in d variables of an integral domain. Then,*

$$h_r(x_1, \dots, x_d) = \sum_{\ell=1}^d x_\ell^{r+d-1} \prod_{\substack{m=1 \\ m \neq \ell}}^d \frac{1}{x_\ell - x_m}.$$

This theorem was proven by Louck-Biedenharn [43, Appendix A, p. 2400] and Cornelius [11]. The proof we present here is new and elementary. The idea of the proof is to establish the theorem as a reformulation of Jacobi's identity (Theorem A.1).

Proof. Substituting $x \rightarrow 1/t$ in Equation (A.1) gives:

$$\prod_{i=1}^d \frac{t}{1 - a_i t} = \sum_{i=1}^d \frac{t}{1 - a_i t} \prod_{\substack{j=1 \\ j \neq i}}^d \frac{1}{a_i - a_j}.$$

Expanding $1/(1 - a_i t)$ in both side of the equation gives

$$\begin{aligned} t^d \sum_r h_r(a_1, \dots, a_d) t^r &= t \sum_{i=1}^d \sum_k a_i^k t^k \prod_{\substack{j=1 \\ j \neq i}}^d \frac{1}{a_i - a_j} \\ \sum_{r=0}^{\infty} h_r(a_1, \dots, a_d) t^{r+d-1} &= \sum_{k=0}^{\infty} \left(\sum_i a_i^k \prod_{\substack{j=1 \\ j \neq i}}^d \frac{1}{a_i - a_j} \right) t^k. \end{aligned}$$

Comparing terms of the same degree in t , we get the final expression of Lemma 1.10:

$$h_r(a_1, \dots, a_d) = \sum_{i=1}^d a_i^{r+d-1} \prod_{\substack{j=1 \\ j \neq i}}^d \frac{1}{a_i - a_j}. \quad (\text{A.5})$$

□

B The Euler Characteristic as the Degree of the Top Chern Class

The purpose of this section is to explain from different points of view why the Euler characteristic is the degree of the top Chern class. Traditionally, this statement is seen as a generalization of the Poincaré-Hopf Theorem that asserts that the total degree of a vector field defined on a smooth manifold M is the Euler characteristic of M . This statement can also be seen as a generalization of the Gauss-Bonnet-Chern Theorem (which is itself is a consequence of Poincaré-Hopf Theorem). Here we will explore three different algebraic approaches. The first one relies on Leftschetz fixed point theorem. The second one is an application of the Hirzebruch-Riemann-Roch Theorem. The third one is an algebraic version of the Poincaré-Hopf Theorem using the interpretation of Chern classes as related to the class of some degenerated loci as discussed in Chapter 3 of Fulton.

B.1 Lefschetz fixed point theorem and the self-intersection number

Theorem B.1. *Let M be a closed oriented manifold of even dimension, then*

$$\int_M e(TM) = \chi(M),$$

where $e(TM)$ is the Euler class of M .

Consider the product $M \times M$ and note that the diagonal Δ is isomorphic to M . The normal bundle $N_{\Delta|M \times M}$ to the diagonal Δ in $M \times M$ is isomorphic to the tangent bundle T_Δ of Δ . The Poincaré dual of a closed oriented submanifold is given by Thom's class in a tubular neighborhood. The Thom class restricted to the zero section of the bundle is by definition the Euler class. It follows that the self-intersection number of the diagonal Δ in $M \times M$ is the Euler number of M .

$$\int_M e(TM) = \int_M \Delta \cdot \Delta.$$

But the self-intersection of the diagonal Δ in $M \times M$ is also the Euler characteristic of M by the Lefschetz fixed point theorem.

Theorem B.2 (Lefschetz fixed point theorem). *Let M be a compact smooth manifold of dimension m and $f : M \rightarrow M$ a smooth map. We define the Lefschetz number of f as*

$$L(f) = \sum_{k=0}^m (-1)^k \operatorname{tr}_k \left(f^* | H^k(M) \right), \quad f^* : H^k(M) \rightarrow H^k(M).$$

Then

$$L(f) = \int_{M \times M} \Gamma_f \cdot \Delta.$$

where Γ_f is the graph of f . Thus $L(f)$ is the number of fixed points counted with multiplicity.

When f is the identity, the Lefschetz trace reduces to the Euler characteristic and the left-hand-side is the self-intersection of the diagonal. The excess intersection formula also ensures that the self-intersection number of Δ in $M \times M$ is the degree of the top Chern class of its normal bundle (which is isomorphic to the tangent bundle of M):

$$\chi(M) = L(\operatorname{Id}) = \int_{M \times M} \Delta \cdot \Delta = \int c(N_\Delta) \cap [\Delta] = \int c(TM) \cap [M] = \int c(M).$$

See also [Fulton, Example 8.1.12] on page 136.

B.2 Hirzebruch-Riemann-Roch

This section follows D. Rössler [?] and [31, Example 18.3.7, p. 363]. First, we recall a lemma of Borel-Serre that is instrumental in the proof of the Hirzebruch-Riemann-Roch Theorem [?].

Lemma B.3 (Borel-Serre (See [31, Example 3.2.5, p. 57])). *Let E be a vector bundle of rank r .*

Then

$$\text{ch}\left(\sum_{p=0}^r (-1)^p \bigwedge^p E^\vee\right) = c_r(E) \text{td}(E)^{-1}.$$

Proof. Borel and Serre [?, Lemma 18, p. 128]. □

Multiplying both sides of the above equation by $\text{td}(E)$ and specializing to $E = TX$ (hence $E^\vee = TX^\vee := \Omega_X$) for X a nonsingular variety of dimension r , gives:

$$\text{ch}\left(\sum_{p=0}^r (-1)^p \bigwedge^p \Omega_X\right) \text{td}(TX) = c_r(TX)$$

Using the additive properties of the Chern character, followed by the Hirzebruch-Riemann-Roch theorem, we rewrite the left hand side of the previous equation as:

$$\text{ch}\left(\sum_{p=0}^r (-1)^p \bigwedge^p \Omega_X\right) \text{td}(TX) = \sum_{\ell, k} (-1)^{\ell+k} \text{rk}(H^\ell(X, \Omega_X^k))$$

Finally, we apply Hodge decomposition to conclude

$$\text{ch}\left(\sum_{p=0}^r (-1)^p \bigwedge^p \Omega_X\right) \text{td}(TX) = \sum_{\ell, k} (-1)^{\ell+k} \text{rk}(H^\ell(X, \Omega_X^k)) = \sum_{\ell} (-1)^\ell H^k(X) = \chi(X).$$

Hence, since $\int c(X) = \int c(TX) \cap [X] = \int_X c_r(TX)$, we get

$$\int c(X) = \int_X c_r(TX) = \chi(X).$$

B.3 Zero-scheme of a section of the tangent bundle

Theorem B.4 ([31], p. 61, Example 3.2.16). *Let E be a vector bundle of rank r on X , s a section of E , and Z the zero-scheme of s . If X is purely n -dimensional and s is a regular section, then Z is purely $(n - r)$ -dimensional, and*

$$[Z] = c_r(E) \cap [X].$$

In particular, if E is the tangent bundle TX of X , s is a vector bundle whose isolated singularities define the class $[Z]$. Hence the degree of the top Chern class gives the the index of s , which is the Euler characteristic of M by the Poincaré-Hopf Theorem:

$$\int c_r(TX) \cap [X] = \chi(X).$$

Note that $c_r(TX) \cap [X]$ is also the self-intersection of the diagonal Δ in $M \times M$ since the normal bundle of Δ in $M \times M$ is isomorphic to the tangent bundle of M . It follows that we can also justify the identity with $\chi(X)$ by using the Lefschetz fixed point theorem. The self-intersection can also be used together with Serre's intersection formula to prove the equality with the Euler characteristic. One can also argue this using the Hirzebruch-Riemann-Roch theorem and the Hodge decomposition theorem.

C Basic Notions

The local ring of a subvariety S of X is denoted $\mathcal{O}_{X,S}$, its maximal ideal is $\mathcal{M}_{X,S}$ and the quotient field is the residue field $\kappa(V) = \mathcal{O}_{X,S}/\mathcal{M}_{X,S}$. The local ring $\mathcal{O}_{X,S}$ is the stalk of the structure sheaf of X at the generic point η_S of S and $\kappa(S)$ is the function field of S . If S is a divisor, $\mathcal{O}_{X,S}$ is a one dimensional local domain. In case X is nonsingular along S , $\mathcal{O}_{X,S}$ is a discrete valuation ring and the order of vanishing is given by the usual valuation.

C.1 Fiber types, dual graphs, Kodaira symbols

Definition C.1 (Algebraic cycle). An algebraic cycle of a Noetherian scheme X is a finite formal sum $\sum_i n_i V_i$ of subvarieties V_i with integer coefficients n_i . If all the subvarieties V_i have the same dimension d , the cycle is called a d -cycle. The free group generated by subvarieties of dimension d is denoted $Z_d(X)$. The group of all cycles, denoted $Z(X) = \bigoplus_d Z_d(X)$, is the free group generated by subvarieties of X .

Definition C.2 (Degree of a zero-cycle [31, Chapter 1, Definition 1.4, p. 13]). Let X be a complete scheme. The *degree* of a zero-cycle $\sum n_i p_i$ of X is

$$\deg(\sum_i n_i p_i) = \sum_i n_i [\kappa(p_i) : k],$$

where $[\kappa(p_i) : k]$ is the degree of the field extension $\kappa(p_i) \rightarrow k$.

Let Θ be an algebraic one-cycle with irreducible decomposition $\Theta = \sum_i m_i \Theta_i$. We denote by $\Theta_i \cdot \Theta_j$ the zero-cycle defined by the intersection of Θ_i and Θ_j for $i \neq j$.

Definition C.3 (n -points, tree). A n -point of an algebraic one-cycle Θ is a point in $\bigcup_i \Theta_i$, which belongs to exactly n distinct irreducible components Θ_i . An algebraic one-cycle Θ is said to be a *tree* if it does not have n -points for $n > 2$. Two curves intersect transversally if their intersection consists of isolated reduced closed points.

Following Kodaira [38], we introduce the following definition:

Definition C.4 (Fiber type). By the *type* of an algebraic one-cycle $\Theta \in Z_1(X)$ with irreducible decomposition $\Theta = \sum_i m_i \Theta_i$, we mean the isomorphism class of each irreducible curve Θ_i , together with the topological structure of the reduced polyhedron $\sum \Theta_i$ (that is the collection of zero-cycles $\Theta_i \cdot \Theta_j$ ($i \neq j$)), and the homology class of $\Theta = \sum_i m_i \Theta_i$ in the Chow group $A_1(X)$.

Example C.5. For instance, $\Theta_1 \cdot \Theta_2 = 2p_1 + 3p_2$ indicates that the two curves Θ_1 and Θ_2 meet at two points p_1 and p_2 with respective intersection multiplicity 2 and 3.

Definition C.6 (Dual graph). To an algebraic one-cycle Θ with irreducible decomposition $\Theta = \sum_i m_i \Theta_i$, we associate a weighted graph (called the *dual graph* of Θ) such that:

- The vertices are the irreducible components of the fiber.
- The weight of a vertex corresponding to the irreducible component Θ_i is its multiplicity m_i . When the multiplicity is one, it can be omitted.

- The vertices corresponding to the irreducible components Θ_i and Θ_j ($i \neq j$) are connected by $\hat{\Theta}_{i,j} = \deg(\Theta_i \cdot \Theta_j)$ edges.

Definition C.7 (Kodaira symbols, See [38]). Kodaira has introduced the following symbols characterizing the type of one-cycles appearing in the study of minimal elliptic surfaces. See Table 4 for a visualization of these fibers.

1. Type I_0 : a smooth curve of genus 1.
2. Type I_1 : an irreducible nodal rational curve.
3. Type II : an irreducible cuspidal rational curve.
4. Type I_2 : $\Theta = \Theta_1 + \Theta_2$ and $\Theta_1 \cdot \Theta_2 = p_1 + p_2$: two smooth rational curves intersecting transversally at two distinct points p_1 and p_2 . The dual graph of I_2 is \tilde{A}_1 .
5. Type III : $\Theta = \Theta_1 + \Theta_2$ and $\Theta_1 \cdot \Theta_2 = 2p$: two smooth rational curves intersecting at a double point. Its dual graph is \tilde{A}_1 .
6. Type IV : $\Theta = \Theta_1 + \Theta_2 + \Theta_3$ and $\Theta_1 \cdot \Theta_2 = \Theta_1 \cdot \Theta_3 = \Theta_2 \cdot \Theta_3 = p$: a 3-star composed of smooth rational curves. Its dual graph is \tilde{A}_2 .
7. Type I_n ($n \geq 3$): $\Theta = \Theta_0 + \dots + \Theta_n$ with $\Theta_i \cdot \Theta_{i+1} = p_i$ $i = 0, \dots, n-1$ and $\Theta_n \cdot \Theta_0 = p_n$. Its dual graph is the affine Dynkin diagram \tilde{A}_{n-1} .
8. Type I_n^* ($n \geq 0$): $\Theta = \Theta_0 + \Theta_1 + 2\Theta_2 + \dots + 2\Theta_{n+2} + \Theta_{n+3} + \Theta_{n+4}$, with $\Theta_i \cdot \Theta_{i+1} = p_i$ ($i = 1, \dots, n+2$), $\Theta_0 \cdot \Theta_2 = p_0$, $\Theta_{n+4} \cdot \Theta_{n+2} = p_{n+4}$. The dual graph the affine Dynkin diagram \tilde{D}_{4+n} .
9. Type IV^* : $\Theta = \Theta_0 + \Theta_1 + 2\Theta_2 + 2\Theta_3 + 3\Theta_4 + 2\Theta_5 + \Theta_6$ with $\Theta_i \cdot \Theta_{i+1} = p_i$ ($i = 3, \dots, 6$), $\Theta_1 \cdot \Theta_3 = p_1$, $\Theta_0 \cdot \Theta_2 = p_0$, $\Theta_2 \cdot \Theta_4 = p_2$. The dual graph is the affine Dynkin diagram \tilde{E}_6 .
10. Type III^* : $\Theta = \Theta_0 + 2\Theta_1 + 2\Theta_2 + 3\Theta_3 + 4\Theta_4 + 3\Theta_5 + 2\Theta_6 + \Theta_7$ with $\Theta_i \cdot \Theta_{i+1} = p_i$ ($i = 3, \dots, 6$), $\Theta_1 \cdot \Theta_3 = p_1$, $\Theta_0 \cdot \Theta_1 = p_0$, $\Theta_2 \cdot \Theta_4 = p_2$. The dual graph is the affine Dynkin diagram \tilde{E}_7 .
11. Type II^* : $\Theta = 2\Theta_1 + 3\Theta_2 + 4\Theta_3 + 6\Theta_4 + 5\Theta_5 + 4\Theta_6 + 3\Theta_7 + 2\Theta_8 + \Theta_0$, with $\Theta_i \cdot \Theta_{i+1} = p_i$ ($i = 3, \dots, 7$), $\Theta_1 \cdot \Theta_3 = p_1$, $\Theta_8 \cdot \Theta_0 = p_8$, and $\Theta_2 \cdot \Theta_4 = p_2$. The dual graph the affine Dynkin diagram \tilde{E}_8 .

C.2 Elliptic fibrations, generic versus geometric fibers

Definition C.8 (Elliptic fibrations). A surjective proper morphism $\varphi : Y \longrightarrow B$ between two algebraic varieties Y and B is called an elliptic fibration if the generic fiber of φ is a smooth projective curve of genus one and φ has a rational section. When B is a curve, Y is called an elliptic surface. When B is a surface, Y is said to be an elliptic threefold. In general, if B has dimension $n-1$, Y is called an elliptic n -fold.

The locus of singular fibers of φ is called its discriminant locus of φ and is denoted $\Delta(\varphi)$ or simply Δ when the context is clear. If the base B is smooth, the discriminant locus is a divisor [18]. The singular fibers of a minimal elliptic surface have been classified by Kodaira and Néron. The

dual graphs of these geometric fibers are affine Dynkin diagrams. We denote these singular fibers by their Kodaira symbols as described in Definition C.7 and presented in Table 4.

The language of schemes streamlines many notions in the study of fibrations. We review some basic definitions.

Definition C.9 (Fiber over a point). Let $\varphi : Y \longrightarrow B$ be a morphism of schemes. For any $p \in B$, the fiber over p is denoted Y_p and defined using a fibral product⁵ as

$$Y_p = Y \times_B \text{Spec } \kappa(p).$$

The first projection $Y_p \longrightarrow Y$ induces an homeomorphism from Y_p onto $f^{-1}(p)$ [42, §3.1 Proposition 1.16]. The second projection gives Y_p the structure of a scheme over the residue field $\kappa(p)$.

If p is not a closed point⁶, the residue field $\kappa(p)$ is not necessarily algebraically closed. Certain components of Y_p could be $\kappa(p)$ -irreducible (i.e. irreducible when defined over $\kappa(p)$) while they become reducible after an appropriate field extension. An irreducible scheme over a field k is said to be *geometrically irreducible* when it stays irreducible after any field extension. The most refined description of the fiber Y_p is always the one corresponding to the algebraic closure $\overline{\kappa(p)}$ of $\kappa(p)$. This motivates the following definition.

Definition C.10. The geometric fiber over p is the fiber $Y_p \times_{\kappa(p)} \overline{\kappa(p)}$, the fiber Y_p after the base change induced by the field extension $\kappa(p) \rightarrow \overline{\kappa(p)}$ to the algebraic closure of $\kappa(p)$.

By construction, a geometric fiber is always composed of geometrically irreducible components.

Definition C.11. We say that the type of a fiber Y_p is *geometric* if it does not change after a field extension.

Remark C.12. To emphasize the difference between the fiber Y_p and its geometric fiber, we will refer to the fiber Y_p (defined with respect to the residue field $\kappa(p)$) as the *arithmetic fiber*.

For an elliptic n -fold, the Kodaira fibers are also the *geometric generic fibers* of the irreducible components of the reduced discriminant locus. While the dual graph of a Kodaira fiber is an affine Dynkin diagram of type \tilde{A}_k , \tilde{D}_{4+k} , \tilde{E}_6 , \tilde{E}_7 , or \tilde{E}_8 , the dual graph of the generic (arithmetic) fiber itself can also be a twisted Dynkin diagram of type \tilde{B}_{3+k}^t , \tilde{C}_{2+k}^t , \tilde{G}_2^t , or \tilde{F}_4^t . This is reviewed in Tables 2 and 3. These dual graphs are not geometric in the sense that after an appropriate base change they becomes \tilde{D}_{4+n} , \tilde{A}_{2+2k} or \tilde{A}_{1+2k} , and \tilde{E}_6 respectively. The Kodaira fibers of the following type never need a field extension:

$$\text{I}_1, \text{II}, \text{III}, \text{III}^*, \text{and } \text{II}^*.$$

The remaining Kodaira fibers (IV , $\text{I}_{n>1}$, I_n^* , and IV^*) can come from fibers Y_p whose types are not geometric and require at least a field extension of degree 2 to describe a fiber with a geometric type. When the fiber Y_p has a geometric type, the type of the fiber is said to be *split*. Otherwise, the type of Y_p is said to be non-split. When that is the case we mark the fiber with an “ns” superscript:

⁵Given three sets (A_1 , A_2 , and S) and two maps $\varphi_1 : A_1 \rightarrow B$ and $\varphi_2 : A_2 \rightarrow B$, we define the fibral product $A_1 \times_S A_2$ as the subset of $A_1 \times A_2$ composed of couples (a_1, a_2) such that $\varphi_1(a_1) = \varphi_2(a_2)$.

⁶For example, if p is the generic point of a subvariety of B .

IV^{ns} , I_n^{ns} , I_n^{*ns} , ($n \geq 2$) and IV^{*ns} . When a field extension is not needed, the fibers are marked with an “s” superscript (“split”): IV^s , I_n^s , I_n^{*s} , ($n \geq 2$) and IV^{*s} . The fiber of type I_0^* can be split, semi-split, or non-split if the Kodaira types require at field extension of degree 3, 2, or 1. The corresponding dual graphs are respectively \tilde{G}_2^t , \tilde{B}_3^t , and \tilde{D}_4 .

C.3 Weierstrass models and Deligne’s formulaire

In this section, we follow the notation of [13]. Let \mathcal{L} be a line bundle over a normal quasi-projective variety B . We define the projective bundle (of lines):

$$\pi : X_0 = \mathbb{P}_B[\mathcal{O}_B \oplus \mathcal{L}^{\otimes 2} \oplus \mathcal{L}^{\otimes 3}] \longrightarrow B. \quad (C.1)$$

The relative projective coordinates of X_0 over B are denoted $[z : x : y]$, where z , x , and y are defined respectively by the natural injection of \mathcal{O}_B , $\mathcal{L}^{\otimes 2}$, and $\mathcal{L}^{\otimes 3}$ into $\mathcal{O}_B \oplus \mathcal{L}^{\otimes 2} \oplus \mathcal{L}^{\otimes 3}$. Hence, z is a section of $\mathcal{O}_{X_0}(1)$, x is a section of $\mathcal{O}_{X_0}(1) \otimes \pi^* \mathcal{L}^{\otimes 2}$, and y is a section of $\mathcal{O}_{X_0}(1) \otimes \pi^* \mathcal{L}^{\otimes 3}$.

Definition C.13. A Weierstrass model is an elliptic fibration $\varphi : Y \rightarrow B$ cut out by the zero locus of a section of the line bundle $\mathcal{O}(3) \otimes \pi^* \mathcal{L}^{\otimes 6}$ in X_0 .

The most general Weierstrass equation is written in the notation of Tate as:

$$y^2 z + a_1 x y z + a_3 y z^2 - (x^3 + a_2 x^2 z + a_4 x z^2 + a_6 z^3) = 0, \quad (C.2)$$

where a_i is a section of $\pi^* \mathcal{L}^{\otimes i}$. The line bundle \mathcal{L} is called the *fundamental line bundle* of the Weierstrass model $\varphi : Y \rightarrow B$. It can be defined directly from Y as $\mathcal{L} = R^1 \varphi_* Y$. Following Tate and Deligne, we introduce the following quantities

$$\begin{cases} b_2 &= a_1^2 + 4a_2 \\ b_4 &= a_1 a_3 + 2a_4 \\ b_6 &= a_3^2 + 4a_6 \\ b_8 &= a_1^2 a_6 - a_1 a_3 a_4 + 4a_2 a_6 + a_2 a_3^2 - a_4^2 \\ c_4 &= b_2^2 - 24b_4 \\ c_6 &= -b_2^3 + 36b_2 b_4 - 216b_6 \\ \Delta &= -b_2^2 b_8 - 8b_4^3 - 27b_6^2 + 9b_2 b_4 b_6 \\ j &= c_4^3 / \Delta \end{cases} \quad (C.3)$$

The b_i ($i = 2, 3, 4, 6$) and c_i ($i = 4, 6$) are sections of $\pi^* \mathcal{L}^{\otimes i}$. The discriminant Δ is a section of $\pi^* \mathcal{L}^{\otimes 12}$. They satisfy the two relations

$$1728\Delta = c_4^3 - c_6^2, \quad 4b_8 = b_2 b_6 - b_4^2. \quad (C.4)$$

Completing the square in y gives

$$zy^2 = x^3 + \frac{1}{4}b_2 x^2 + \frac{1}{2}b_4 x + \frac{1}{4}b_6. \quad (C.5)$$

Completing the cube in x gives the short form of the Weierstrass equation

$$zy^2 = x^3 - \frac{1}{48}c_4xz^2 - \frac{1}{864}c_6z^3. \quad (\text{C.6})$$

C.4 Tate's algorithm

Let R be a complete discrete valuation ring with valuation v , uniformizing parameter s , and perfect residue field $\kappa = R/(s)$. We are interested in the case where κ has characteristic zero.

Remark C.14. We recall that a discrete valuation ring has only three ideals, the zero ideal, the ring itself, and the principal ideal sR . It follows that the scheme $\text{Spec}(R)$ has only two points⁷: the generic point (defined by the zero ideal) and the closed point (defined by the principal ideal sR).

Let E/R be an elliptic curve over R with Weierstrass equation

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6, \quad a_i \in R.$$

The generic fiber is a regular elliptic curve. After a resolution of singularities, we have a regular model \mathcal{E} over R and the *special fiber* is the fiber over the closed point $\text{Spec } R/(s)$.

Tate's algorithm determines the type of the geometric special fibers over the closed point of $\text{Spec}(R)$ by manipulating the valuations of the coefficients and the discriminant and the arithmetic properties of some auxiliary polynomials. The type of the *geometric fiber* is given by its Kodaira's symbol. The special fiber becomes geometric after a quadratic or a cubic field extension κ'/κ . Keeping track of the field extension used gives a classification of the special fiber as a κ -scheme—this is what we call the arithmetic fiber. The information on the required field extension needed to have geometrically irreducible components is already carefully encoded in Tate's original algorithm, as it is needed to compute the local index c .

Tate's algorithm consists of the following eleven steps (see [60] , [57, §IV.9], [16]).

Step 1. $v(\Delta) = 0 \implies \text{I}_0$.

Step 2. If $v(\Delta) \geq 1$, change coordinates so that $v(a_3) \geq 1$, $v(a_4) \geq 1$, and $v(a_6) \geq 1$.

If $v(b_2) = 0$, the type is $\text{I}_{v(\Delta)}$. To have a fiber with geometric irreducible components, it is enough to work in the splitting field κ' of the following polynomial of $\kappa[T]$:

$$T^2 + a_1T - a_2.$$

The discriminant of this quadric is b_2 . If b_2 is a square in κ , then $\kappa' = \kappa$, otherwise $\kappa' \neq \kappa$:

$$(a) \ \kappa' = \kappa \implies \text{I}_n^s \quad (b) \ \kappa' \neq \kappa \implies \text{I}_n^{\text{ns}}$$

Step 3. $v(b_2) \geq 1$, $v(a_3) \geq 1$, $v(a_4) \geq 1$, and $v(a_6) = 1 \implies \text{II}$.

Step 4. $v(b_2) \geq 1$, $v(a_3) \geq 1$, $v(a_4) = 1$, and $v(a_6) \geq 2 \implies \text{III}$.

⁷As usual we take the convention in which the ring itself is not a prime ideal.

Step 5. $v(b_2) \geq 1, v(a_3) \geq 1, v(a_4) \geq 2, v(a_6) \geq 2$, and $v(b_6) = 2 \implies \text{IV}$.

The fiber has geometric irreducible components over the splitting field κ' of the polynomial

$$T^2 + a_{3,1}T - a_{6,2}$$

Its discriminant is $b_{6,2}$. If $b_{6,2}$ is a square in κ , then $\kappa' = \kappa$ otherwise $\kappa' \neq \kappa$.

(a) $\kappa' = \kappa \implies \text{IV}^s$ (b) $\kappa' \neq \kappa \implies \text{IV}^{ns}$

Step 6. $v(b_2) \geq 1, v(a_3) \geq 1, v(a_4) \geq 2, v(a_6) \geq 3, v(b_6) \geq 3, v(b_8) \geq 3$. Then make a change of coordinates such that $v(a_1) \geq 1, v(a_2) \geq 1, v(a_3) \geq 2, v(a_4) \geq 2$, and $v(a_6) \geq 3$. Let

$$P(T) = T^3 + a_{2,1}T^2 + a_{4,2}T + a_{6,3}$$

If $P(T)$ is a separable polynomial in κ , that is if $P(T)$ has three distinct roots in a field extension of κ , then the type is I_0^* . The geometric fiber is defined over the splitting field κ' of $P(T)$ in κ . The type of the special fiber before to go to the splitting field depends on the degree of the field extension $\kappa' \rightarrow \kappa$:

- $[\kappa' : \kappa] = 3 \implies \text{I}_0^{*ns}$ with dual graph \tilde{G}_2^t .
- $[\kappa' : \kappa] = 2 \implies \text{I}_0^{*ss}$ with dual graph \tilde{B}_3^t .
- $[\kappa' : \kappa] = 1 \implies \text{I}_0^{*s}$ with dual graph \tilde{D}_4 .

where “ns”, “ss”, and “s” stand respectively for “non-split”, “semi-split”, and “split”. In the notation of Liu, these fibers are respectively $\text{I}_{0,3}^*$, $\text{I}_{0,2}^*$, and I_0^* .

Step 7. If $P(T)$ has a double root, then the type is I_n^* .

Make a change of coordinates such that the double root is at the origin. Then

$$v(a_1) \geq 1, \quad v(a_2) = 1, \quad v(a_3) \geq 2, \quad v(a_4) \geq 3, \quad , \quad v(a_6) \geq 4, \quad \text{and} \quad v(\Delta) = n + 6 \quad (n \geq 1).$$

Let

$$Q(T) = T^2 + a_{3,2+\lfloor n/2 \rfloor}T - a_{6,3+\lfloor n/2 \rfloor}.$$

If $Q(T)$ has distinct rational roots in κ , we have I_n^{*s} otherwise we have I_n^{*ns} .

Step 8. If $P(T)$ has a triple root, change coordinates such that the triple root is zero. Then $v(a_1) \geq 1, v(a_2) \geq 2, v(a_3) \geq 2, v(a_4) \geq 3, v(a_6) \geq 4$.

Let

$$Q(T) = T^2 + a_{3,2}T - a_{6,4}$$

If Q has two distinct roots ($v(b_6) = 4$ or equivalently $v(\Delta) = 8$) the type is IV^* .

The split type depends on the rationality of the roots. If $b_{6,4}$ is a perfect square modulo s , the fiber is IV^{*s} , otherwise the fiber is IV^{*ns} .

The split form can be enforced with $v(a_6) \geq 5$ and hence $v(a_3) = 2$ to ensure that $v(b_6) = 4$.

Step 9. If Q has a double root, we change coordinates so that the double root is at the origin. Then: $v(a_1) \geq 1, \quad v(a_2) \geq 2, \quad v(a_3) \geq 3, \quad v(a_4) = 3, \quad v(a_6) \geq 5 \implies \text{type III}^*$.

Step 10. $v(a_1) \geq 1$, $v(a_2) \geq 2$, $v(a_3) \geq 3$, $v(a_4) \geq 4$, $v(a_6) = 5 \implies$ type II*.

Step 11. Else $v(a_i) \geq i$ and the equation is not minimal. Divides all the a_i by s^i and start again with the new equation.

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